Chapter 10

Further Techniques and Applications of Integration

Some simple geometric problems require advanced methods of integration.

Besides the basic methods of integration associated with reversing the differentiation rules, there are special methods for integrands of particular forms. Using these methods, we can solve some interesting length and area problems.

10.1 Trigonometric Integrals

The key to evaluating many integrals is a trigonometric identity or substitution.

The integrals treated in this section fall into two groups. First, there are some purely trigonometric integrals that can be evaluated using trigonometric identities. Second, there are integrals involving quadratic functions and their square roots which can be evaluated using trigonometric substitutions.

We begin by considering integrals of the form

$$\int \sin^m x \cos^n x\, dx,$$

where m and n are integers. The case $n = 1$ is easy, for if we let $u = \sin x$, we find

$$\int \sin^m x \cos x\, dx = \int u^m\, du = \frac{u^{m+1}}{m+1} + C = \frac{\sin^{m+1}(x)}{m+1} + C$$

(or $\ln|\sin x| + C$, if $m = -1$). The case $m = 1$ is similar:

$$\int \sin x \cos^n x\, dx = -\frac{\cos^{n+1}(x)}{n+1} + C$$

(or $-\ln|\cos x| + C$, if $n = -1$). If either m or n is odd, we can use the identity $\sin^2 x + \cos^2 x = 1$ to reduce the integral to one of the types just treated.

Example 1 Evaluate $\int \sin^2 x \cos^3 x\, dx$.

Solution $\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x \cos^2 x \cos x\, dx = \int (\sin^2 x)(1 - \sin^2 x)\cos x\, dx$, which can be integrated by the substitution $u = \sin x$. We get

$$\int u^2(1-u^2)\, du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C. \blacktriangle$$

If $m = 2k$ and $n = 2l$ are both even, we can use the half-angle formulas $\sin^2 x = (1 - \cos 2x)/2$ and $\cos^2 x = (1 + \cos 2x)/2$ to write

$$\int \sin^{2k}x \cos^{2l}x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)^k \left(\frac{1+\cos 2x}{2}\right)^l dx$$

$$= \frac{1}{2}\int \left(\frac{1-\cos y}{2}\right)^k \left(\frac{1+\cos y}{2}\right)^l dy,$$

where $y = 2x$. Multiplying this out, we are faced with a sum of integrals of the form $\int \cos^m y\,dy$, with m ranging from zero to $k + l$. The integrals for odd m can be handled by the previous method; to those with even m we apply the half-angle formula once again. The whole process is repeated as often as necessary until everything is integrated.

Example 2 Evaluate $\int \sin^2 x \cos^2 x\,dx$.

Solution

$$\int \sin^2 x \cos^2 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{4}\int (1 - \cos^2 2x)\,dx = \frac{x}{4} - \frac{1}{4}\int \cos^2 2x\,dx$$

$$= \frac{x}{4} - \frac{1}{4}\int \frac{1+\cos 4x}{2}\,dx = \frac{x}{4} - \frac{x}{8} - \frac{1}{8}\int \cos 4x\,dx$$

$$= \frac{x}{8} - \frac{\sin 4x}{32} + C. \blacktriangle$$

Trigonometric Integrals

To evaluate $\int \sin^m x \cos^n x\,dx$:

1. If m is odd, write $m = 2k + 1$, and

$$\int \sin^m x \cos^n x\,dx = \int \sin^{2k}x \cos^n x \sin x\,dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \sin x\,dx.$$

Now integrate by substituting $u = \cos x$.

2. If n is odd, write $n = 2l + 1$, and

$$\int \sin^m x \cos^n x\,dx = \int \sin^m x \cos^{2l}x \cos x\,dx$$

$$= \int \sin^m x (1 - \sin^2 x)^l \cos x\,dx.$$

Now integrate by substituting $u = \sin x$.

3. (a) If m and n are even, write $m = 2k$ and $n = 2l$ and

$$\int \sin^{2k}x \cos^{2l}x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)^k \left(\frac{1+\cos 2x}{2}\right)^l dx.$$

Substitute $y = 2x$. Expand and apply step 2 to the odd powers of $\cos y$.
(b) Apply step 3(a) to the even powers of $\cos y$ and continue until the integration is completed.

Example 3 Evaluate: (a) $\int_0^{2\pi}\sin^4 x\cos^2 x\,dx$ (b) $\int(\sin^2 x+\sin^3 x\cos^2 x)\,dx$. (c) $\int\tan^3\theta\sec^3\theta\,d\theta$.

Solution (a) Substitute $\sin^2 x=(1-\cos 2x)/2$ and $\cos^2 x=(1+\cos 2x)/2$ to get

$$\begin{aligned}\int\sin^4 x\cos^2 x\,dx&=\int\frac{(1-\cos 2x)^2}{4}\,\frac{(1+\cos 2x)}{2}\,dx\\&=\frac{1}{8}\int(1-2\cos 2x+\cos^2 2x)(1+\cos 2x)\,dx\\&=\frac{1}{8}\int(1-\cos 2x-\cos^2 2x+\cos^3 2x)\,dx\\&=\frac{1}{16}\int(1-\cos y-\cos^2 y+\cos^3 y)\,dy,\end{aligned}$$

where $y=2x$. Integrating the last two terms gives

$$\int\cos^2 y\,dy=\int\frac{(1+\cos 2y)}{2}\,dy=\frac{y}{2}+\frac{\sin 2y}{4}+C$$

and

$$\int\cos^3 y\,dy=\int(1-\sin^2 y)\cos y\,dy=\sin y-\frac{\sin^3 y}{3}+C.$$

Thus

$$\begin{aligned}\int\sin^4 x\cos^2 x\,dx&=\frac{1}{16}\left(y-\sin y-\frac{y}{2}-\frac{\sin 2y}{4}+\sin y-\frac{\sin^3 y}{3}\right)+C\\&=\frac{1}{16}\left(\frac{y}{2}-\frac{\sin 2y}{4}-\frac{\sin^3 y}{3}\right)+C\\&=\frac{1}{16}\left(x-\frac{\sin 4x}{4}-\frac{\sin^3 2x}{3}\right)+C,\end{aligned}$$

and so $\int_0^{2\pi}\sin^4 x\cos^2 x\,dx=\frac{\pi}{8}$.

(b)
$$\begin{aligned}\int(\sin^2 x+\sin^3 x\cos^2 x)\,dx&=\int\sin^2 x\,dx+\int\sin^3 x\cos^2 x\,dx\\&=\int\left(\frac{1-\cos 2x}{2}\right)dx\\&\quad+\int(1-\cos^2 x)\cos^2 x\sin x\,dx\\&=\frac{x}{2}-\frac{\sin 2x}{4}-\int(1-u^2)u^2\,du\qquad(u=\cos x)\\&=\frac{x}{2}-\frac{\sin 2x}{4}-\frac{\cos^3 x}{3}+\frac{\cos^5 x}{5}+C.\end{aligned}$$

(c) *Method* 1. Rewrite in terms of $\sec\theta$ and its derivative $\tan\theta\sec\theta$:

$$\begin{aligned}\int\tan^3\theta\sec^3\theta\,d\theta&=\int(\tan\theta\sec\theta)(\tan^2\theta\sec^2\theta)\,d\theta\\&=\int(\tan\theta\sec\theta)(\sec^2\theta-1)\sec^2\theta\,d\theta\qquad(1+\tan^2\theta=\sec^2\theta)\\&=\int(u^2-1)u^2\,du\qquad(u=\sec\theta)\\&=\frac{u^5}{5}-\frac{u^3}{3}+C=\frac{\sec^5\theta}{5}-\frac{\sec^3\theta}{3}+C.\end{aligned}$$

Method 2. Convert to sines and cosines:

$$\int \tan^3\theta \sec^3\theta \, d\theta = \int \frac{\sin^3\theta}{\cos^6\theta}\, d\theta = \int \frac{\sin\theta(1-\cos^2\theta)}{\cos^6\theta}\, d\theta$$

$$= -\int \frac{1-u^2}{u^6}\, du \qquad (u = \cos\theta)$$

$$= \frac{u^{-5}}{5} - \frac{u^{-3}}{3} + C = \frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3} + C. \ \blacktriangle$$

Certain other integration problems yield to the use of the addition formulas:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y, \tag{1a}$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \tag{1b}$$

and the product formulas:

$$\sin x \cos y = \tfrac{1}{2}[\sin(x-y) + \sin(x+y)], \tag{2a}$$

$$\sin x \sin y = \tfrac{1}{2}[\cos(x-y) - \cos(x+y)], \tag{2b}$$

$$\cos x \cos y = \tfrac{1}{2}[\cos(x-y) + \cos(x+y)]. \tag{2c}$$

Example 4 Evaluate (a) $\int \sin x \cos 2x\, dx$ and (b) $\int \cos 3x \cos 5x\, dx$.

Solution (a) $\displaystyle\int \sin x \cos 2x\, dx = \frac{1}{2}\int (\sin 3x - \sin x)\, dx = -\frac{\cos 3x}{6} + \frac{\cos x}{2} + C.$

(see product formula (2a)).

(b) $\displaystyle\int \cos 3x \cos 5x\, dx = \frac{1}{2}\int (\cos 8x + \cos 2x)\, dx = \frac{\sin 8x}{16} + \frac{\sin 2x}{4} + C$

(see product formula (2c)). ▲

Example 5 Evaluate $\int \sin ax \sin bx\, dx$, where a and b are constants.

Solution If we use identity (2b), we get

$$\int \sin ax \sin bx\, dx = \frac{1}{2}\int [\cos(a-b)x - \cos(a+b)x]\, dx$$

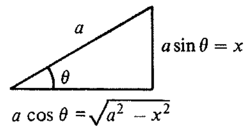
$$= \begin{cases} \dfrac{1}{2}\dfrac{\sin(a-b)x}{a-b} - \dfrac{1}{2}\dfrac{\sin(a+b)x}{a+b} + C & \text{if } a \neq \pm b, \\ \dfrac{x}{2} - \dfrac{1}{4a}\sin 2ax + C, & \text{if } a = b, \\ \dfrac{1}{4a}\sin 2ax - \dfrac{x}{2} + C, & \text{if } a = -b. \end{cases}$$

[The difference between the case $a \neq \pm b$ and the other two should be noted. The first case is "pure oscillation" in that it consists of two sine terms. The others contain the nonoscillating linear term $x/2$, called a *secular term*. This example is related to the phenomena of *resonance*: when an oscillating system is subjected to a sinusoidally varying force, the oscillation will build up indefinitely if the force has the same frequency as the oscillator. See Review Exercise 56, Chapter 8 and the discussion in the last part of Section 12.7, following equation (14).] ▲

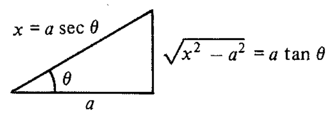
Many integrals containing factors of the form $\sqrt{a^2 \pm x^2}$, $\sqrt{x^2 - a^2}$, or $a^2 + x^2$ can be evaluated or simplified by means of trigonometric substitutions. In order to remember what to substitute, it is useful to draw the appropriate right-angle triangle, as in the following box.

Trigonometric Substitutions

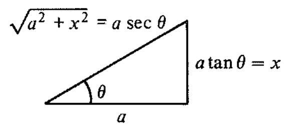
1. If $\sqrt{a^2 - x^2}$ occurs, try $x = a\sin\theta$; then $dx = a\cos\theta\, d\theta$ and $\sqrt{a^2 - x^2} = a\cos\theta$; ($a > 0$ and θ is an acute angle).



2. If $\sqrt{x^2 - a^2}$ occurs, try $x = a\sec\theta$; then $dx = a\tan\theta\sec\theta\, d\theta$ and $\sqrt{x^2 - a^2} = a\tan\theta$.



3. If $\sqrt{a^2 + x^2}$ or $a^2 + x^2$ occurs, try $x = a\tan\theta$; then $dx = a\sec^2\theta\, d\theta$ and $\sqrt{a^2 + x^2} = a\sec\theta$ (one can also use $x = a\sinh\theta$; then $\sqrt{a^2 + x^2} = a\cosh\theta$).



Example 6 Evaluate: (a) $\int \frac{\sqrt{9 - x^2}}{x^2}\, dx$, (b) $\int \frac{dx}{\sqrt{4x^2 - 1}}$.

Solution (a) Let $x = 3\sin\theta$, so $\sqrt{9 - x^2} = 3\cos\theta$. Thus $dx = 3\cos\theta\, d\theta$ and

$$\begin{aligned}
\int \frac{\sqrt{9 - x^2}}{x^2}\, dx &= \int \frac{3\cos\theta}{9\sin^2\theta} 3\cos\theta\, d\theta \\
&= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \int \frac{1 - \sin^2\theta}{\sin^2\theta}\, d\theta \\
&= \int (\csc^2\theta - 1)\, d\theta = -\cot\theta - \theta + C \\
&= -\frac{\sqrt{9 - x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C.
\end{aligned}$$

In the last line, we used the first figure in the preceding box to get the identity $\cot\theta = \sqrt{a^2 - x^2}/x$ with $a = 3$.

(b) Let $x = \frac{1}{2}\sec\theta$, so $dx = \frac{1}{2}\tan\theta\sec\theta\,d\theta$, and $\sqrt{4x^2-1} = \tan\theta$. Thus

$$\int \frac{dx}{\sqrt{4x^2-1}} = \frac{1}{2}\int \frac{\tan\theta\sec\theta}{\tan\theta}\,d\theta = \frac{1}{2}\int \sec\theta\,d\theta.$$

Here is a trick[1] for evaluating $\int \sec\theta$:

$$\int \sec\theta\,d\theta = \int \sec\theta\,\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\,d\theta = \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta$$
$$= \ln|\sec\theta+\tan\theta| + C \qquad (\text{substituting } u = \sec\theta+\tan\theta).$$

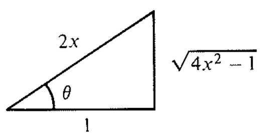


Figure 10.1.1. Geometry of the substitution $x = \frac{1}{2}\sec\theta$.

Thus

$$\int \frac{dx}{\sqrt{4x^2-1}} = \frac{1}{2}\ln\left|2x+\sqrt{4x^2-1}\right| + C$$

(see Fig. 10.1.1).

If you have studied the hyperbolic functions you should note that this integral can also be evaluated by means of the formula $\int[du/\sqrt{u^2-1}] = \cosh^{-1}u + C$. ▲

These examples show that trigonometric substitutions work quite well in the presence of algebraic integrands involving square roots. You should also keep in mind the possibility of a simple algebraic substitution or using the direct integration formulas involving inverse trigonometric and hyperbolic functions.

Example 7 Evaluate:

(a) $\displaystyle\int \frac{x}{\sqrt{4-x^2}}\,dx$; (b) $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx$; (c) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$.

Solution (a) Let $u = 4-x^2$, so $du = -2x\,dx$. Thus

$$\int \frac{x}{\sqrt{4-x^2}}\,dx = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\sqrt{u} + C = -\sqrt{4-x^2} + C$$

(no trigonometric function appears).

(b)
$$\int \frac{1}{\sqrt{x^2-4}}\,dx = \int \frac{du}{\sqrt{u^2-1}} \qquad \left(u = \frac{x}{2}\right)$$
$$= \cosh^{-1}u + C = \cosh^{-1}\left(\frac{x}{2}\right) + C$$
$$= \ln\left(\frac{x}{2} + \sqrt{\frac{x^2}{4}-1}\right) + C \qquad (\text{see p. 396}).$$

You may use the method of Example 6(b) if you are not familiar with hyperbolic functions.

(c) To evaluate $\int(x^2/\sqrt{4-x^2})\,dx$, let $x = 2\sin\theta$; then $dx = 2\cos\theta\,d\theta$, and $\sqrt{4-x^2} = 2\cos\theta$. Thus

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx = \int \frac{4\sin^2\theta}{2\cos\theta}\cdot 2\cos\theta\,d\theta = 4\int \sin^2\theta\,d\theta$$
$$= 4\int \frac{1-\cos 2\theta}{2}\,d\theta = 2\theta - \sin 2\theta + C$$
$$= 2\theta - 2\sin\theta\cos\theta + C.$$

[1] The same trick shows that $\int \operatorname{csch}\theta\,d\theta = -\ln|\operatorname{csch}\theta + \coth\theta| + C$.

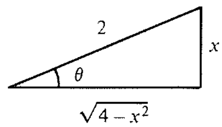


Figure 10.1.2. Geometry of the substitution $x = 2\sin\theta$.

From Fig. 10.1.2 we get

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx = 2\sin^{-1}\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + C$$
$$= 2\sin^{-1}\frac{x}{2} - \frac{1}{2}x\sqrt{4-x^2} + C. \blacktriangle$$

Completing the square can be useful in simplifying integrals involving the expression $ax^2 + bx + c$. The following two examples illustrate the method.

Example 8 Evaluate $\int \frac{dx}{\sqrt{10+4x-x^2}}$.

Solution To complete the square, write $10 + 4x - x^2 = -(x+a)^2 + b$; solving for a and b, we find $a = -2$ and $b = 14$, so $10 + 4x - x^2 = -(x-2)^2 + 14$. Hence

$$\int \frac{dx}{\sqrt{10+4x-x^2}} = \int \frac{dx}{\sqrt{14-(x-2)^2}} = \int \frac{du}{\sqrt{14-u^2}},$$

where $u = x - 2$. This integral is $\sin^{-1}(u/\sqrt{14}) + C$, so our final answer is

$$\sin^{-1}\left(\frac{x-2}{\sqrt{14}}\right) + C. \blacktriangle$$

Completing the Square

If an integral involves $ax^2 + bx + c$, complete the square and then use a trigonometric substitution or some other method to evaluate the integral.

Example 9 Evaluate (a) $\int \frac{dx}{x^2+x+1}$; (b) $\int \frac{dx}{\sqrt{x^2+x+1}}$.

Solution (a)
$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{(x+1/2)^2+3/4}$$
$$= \int \frac{du}{u^2+3/4} \qquad \left(u = x + \frac{1}{2}\right)$$
$$= \frac{1}{\sqrt{3/4}}\tan^{-1}\left(\frac{u}{\sqrt{3/4}}\right) + C$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

(b)
$$\int \frac{dx}{\sqrt{x^2+x+1}} = \int \frac{du}{\sqrt{u^2+3/4}} \qquad \left(u = x + \frac{1}{2}\right)$$
$$= \ln\left|u + \sqrt{u^2+3/4}\right| + C$$
$$= \ln\left|x + \frac{1}{2} + \sqrt{(x+1/2)^2+3/4}\right| + C$$
$$= \ln\left|x + \frac{1}{2} + \sqrt{x^2+x+1}\right| + C. \blacktriangle$$

Applications of the kind encountered in earlier chapters may involve integrals of the type in this section. Here is an example.

Example 10 Find the average value of $\sin^2 x\cos^2 x$ on the interval $[0, 2\pi]$.

Solution By definition, the average value is the integral divided by the length of the interval:

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^2 x\cos^2 x\,dx.$$

By Example 2, $\int\sin^2 x\cos^2 x\,dx = (x/8) - (\sin 4x/32) + C$. Thus

$$\int_0^{2\pi}\sin^2 x\cos^2 x\,dx = \left(\frac{x}{8} - \frac{\sin 4x}{32}\right)\Big|_0^{2\pi} = \frac{\pi}{4},$$

so the average value is $(1/2\pi)\cdot\pi/4 = 1/8$. ▲

Exercises for Section 10.1

Evaluate the integrals in Exercises 1–12.

1. $\int\sin^3 x\cos^3 x\,dx$
2. $\int\sin^2 x\cos^5 x\,dx$
3. $\int_0^{2\pi}\sin^4 t\,dt$
4. $\int_0^{\pi/2}\cos^4 x\sin^2 x\,dx$
5. $\int(\cos 2x - \cos^2 x)\,dx$
6. $\int\cos 2x\sin x\,dx$
7. $\int_0^{\pi/4}\sin^2 x\cos 2x\,dx$
8. $\int_0^{\pi/4}\left(\frac{\sin^2\theta}{\cos^2\theta}\right)d\theta$
9. $\int\sin 4x\sin 2x\,dx$
10. $\int\sin 2\theta\cos 5\theta\,d\theta$
11. $\int_0^{2\pi}\sin 5x\sin 2x\,dx$
12. $\int_{-\pi}^{\pi}\cos 2u\sin\frac{1}{2}u\,du$
13. Evaluate $\int\tan^3 x\sec^3 x\,dx$. [*Hint*: Convert to sines and cosines.]
14. Show that $\int\sin^6 x\,dx = \frac{1}{192}(60x - 48\sin 2x + 4\sin^3 2x + 9\sin 4x) + C$.
15. Evaluate $\int[1/(1+x^2)]\,dx$ (a) as $\tan^{-1}x$ and (b) by the substitution $x = \tan u$. Compare your answers.
16. Evaluate the integral $\int[1/(4+9x^2)]\,dx$ by using the substitutions (a) $x = \frac{2}{3}u$ and (b) $x = \frac{2}{3}\tan\theta$. Compare your answers.

Evaluate the integrals in Exercises 17–28.

17. $\int\frac{\sqrt{x^2-4}}{x}\,dx$
18. $\int\frac{\sqrt{x^2-9}}{x}\,dx$
19. $\int\sqrt{1-u^2}\,du$
20. $\int\sqrt{9-16t^2}\,dt$
21. $\int\frac{s}{\sqrt{4+s^2}}\,ds$
22. $\int\frac{x}{\sqrt{x^2-1}}\,dx$
23. $\int\frac{x^3}{\sqrt{4-x^2}}\,dx$
24. $\int\frac{x^2}{(1+x^2)^{3/2}}\,dx$
25. $\int\frac{dx}{\sqrt{4x^2+x+1}}$
26. $\int\frac{dx}{\sqrt{5-4x-x^2}}$
27. $\int\frac{x\,dx}{\sqrt{3x^2+x-1}}$
28. $\int\sqrt{3+2x-x^2}\,dx$
29. Find the average value of $\cos^n x$ on the interval $[0, 2\pi]$ for $n = 0, 1, 2, 3, 4, 5, 6$.
30. Find the volume of the solid obtained by revolving the region under the graph $y = \sin^2 x$ on $[0, 2\pi]$ about the x axis.
31. Find the center of mass of the region under the graph of $1/\sqrt{x^2+2x+2}$ on $[0, 1]$.
32. A plating company wishes to prepare the bill for a silver plate job of 200 parts. Each part has the shape of the region bounded by $y = \sqrt{x^2 - 9/x^2}$, $y = 0$, $x = 5$.
    (a) Find the area enclosed.
    (b) Assume that all units are centimeters. Only one side of the part is to receive the silver plate. The customer was charged \$25 for 1460 square centimeters previously. How much should the 200 parts cost?
33. The *average power* P for a resistance R and associated current i of period T is
$$P = \frac{1}{T}\int_0^T Ri^2\,dt.$$
That is, P is the average value of the instantaneous power Ri^2 on $[0, T]$. Compute the power for $R = 2.5$, $i = 10\sin(377t)$, $T = 2\pi/377$.
34. The current I in a certain RLC circuit is given by $I(t) = Me^{-\alpha t}[\sin^2(\omega t) + 2\cos(2\omega t)]$. Find the charge Q in coulombs, given by
$$Q(t) = Q_0 + \int_0^t I(s)\,ds.$$
35. The *root mean square current* and *voltage* are
$$I_{\text{rms}} = \left(\frac{1}{T}\int_0^T i^2\,dt\right)^{1/2}, \quad\text{and}$$
$$E_{\text{rms}} = \left(\frac{1}{T}\int_0^T e^2\,dt\right)^{1/2}$$

where $i(t)$ and $e(t)$ are the current through and voltage across a pure resistance R. (The current flowing through R is assumed to be periodic with period T.) Compute these numbers, given that $e(t) = 3 + (1.5)\cos(100t)$ volts, and $i(t) = 1 - 2\sin(100t - \pi/6)$ amperes, which corresponds to period $T = 2\pi/100$.

36. The *average power* $P = (1/T)\int_0^T Ri^2\,dt$ for periodic waveshapes does not in general obey a superposition principle. Two voltage sources e_1 and e_2 may individually supply 5 watts (when the other is dead), but when both sources are present the power can be zero (not 10). Compare $\int_0^T R(i_1 + i_2)^2\,dt$ with $\int_0^T Ri_1^2\,dt + \int_0^T Ri_2^2\,dt$ when $i_1 = I_1\cos(m\omega t + \phi_1)$, $i_2 = I_2\cos(n\omega t + \phi_2)$, $m \neq n$ (m, n positive integers), $T = 2\pi/\omega$, and $R, I_1, I_2, \omega, \phi_1, \phi_2$ are constants.

37. A charged particle is constrained by magnetic fields to move along a straight line, oscillating back and forth from the origin with higher and higher amplitude.
Let $S(t)$ be the directed distance from the origin, and assume that $S(t)$ satisfies the equation
$$[S(t)]^2 S'(t) = t\sin t + \sin^2 t\cos^2 t.$$
(a) Prove $[S(t)]^3 = 3\int_0^t (x\sin x + \sin^2 x\cos^2 x)\,dx$.
(b) Find $S(t)$.
(c) Find all zeros of $S'(t)$ for $t > 1$. Which zeros correspond to times of maximum excursion from the origin?

★38. Show that the integral in Example 5 is a continuous function of b for fixed a and x.

10.2 Partial Fractions

By the method of partial fractions, one can evaluate any integral of the form $\int \frac{P(x)}{Q(x)}\,dx$*, where P and Q are polynomials.*

The integral of a polynomial can be expressed simply by the formula
$$\int (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)\,dx = \frac{a_n x^{n+1}}{n+1} + \frac{a_{n-1}x^n}{n} + \cdots + a_0 x + C,$$
but there is no simple general formula for integrals of *quotients* of polynomials, i.e., for rational functions. There is, however, a general *method* for integrating rational functions, which we shall learn in this section. This method demonstrates clearly the need for evaluating integrals by hand or by a computer program such as MACSYMA, which automatically carries out the procedures to be described in this section, since tables cannot include the infinitely many possible integrals of this type.

One class of rational functions which we can integrate simply are the reciprocal powers. Using the substitution $u = ax + b$, we find that $\int[dx/(ax+b)^n] = \int(du/au^n)$, which is evaluated by the power rule. Thus, we get
$$\int \frac{dx}{(ax+b)^n} = \begin{cases} \dfrac{-1}{a(n-1)(ax+b)^{n-1}} + C, & \text{if } n \neq 1, \\ \dfrac{1}{a}\ln|ax+b| + C, & \text{if } n = 1. \end{cases}$$

More generally, we can integrate any rational function whose denominator can be factored into linear factors. We shall give several examples before presenting the general method.

Example 1 Evaluate $\int \frac{x+1}{(x-1)(x-3)}\,dx$.

Solution We shall try to write
$$\frac{x+1}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3},$$

for constants A and B. To determine them, note that

$$\frac{A}{x-1}+\frac{B}{x-3}=\frac{(A+B)x-3A-B}{(x-1)(x-3)}.$$

Thus, we should choose

$$A+B=1 \quad \text{and} \quad -3A-B=1.$$

Solving, $A=-1$ and $B=2$. Thus,

$$\frac{x+1}{(x-1)(x-3)}=-\frac{1}{x-1}+\frac{2}{x-3},$$

so

$$\int \frac{x+1}{(x-1)(x-3)}\,dx=-\ln|x-1|+2\ln|x-3|+C$$

$$=\ln\left(\frac{|x-3|^2}{|x-1|}\right)+C. \blacktriangle$$

Example 2 Evaluate

(a) $\displaystyle\int \frac{4x^2+2x+3}{(x-2)^2(x+3)}\,dx$; (b) $\displaystyle\int_{-1}^{1} \frac{4x^2+2x+3}{(x-2)^2(x+3)}\,dx.$

Solution (a) As in Example 1, we might expect to decompose the quotient in terms of $1/(x-2)$ and $1/(x+3)$. In fact, we shall see that we can write

$$\frac{4x^2+2x+3}{(x-2)^2(x+3)}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x+3} \tag{1}$$

if we choose the constants A, B, and C suitably. Adding the terms on the right-hand side of equation (1) over the common denominator, we get

$$\frac{A(x-2)(x+3)+B(x+3)+C(x-2)^2}{(x-2)^2(x+3)}.$$

The numerator, when multiplied out, would be a polynomial $a_2x^2+a_1x+a_0$, where the coefficients a_2, a_1, and a_0 depend on A, B, and C. The idea is to choose A, B, and C so that we get the numerator $4x^2+2x+3$ of our integration problem. (Notice that we have exactly three unknowns A, B, and C at our disposal to match the three coefficients in the numerator.)

To choose A, B, and C, it is easiest not to multiply out but simply to write

$$4x^2+2x+3=A(x-2)(x+3)+B(x+3)+C(x-2)^2 \tag{2}$$

and make judicious substitutions for x. For instance, $x=-3$ gives

$$4\cdot 9-2\cdot 3+3=C(-3-2)^2,$$
$$33=25C,$$
$$C=\tfrac{33}{25}.$$

Next, $x=2$ gives

$$4\cdot 4+2\cdot 2+3=B(2+3),$$
$$23=5B,$$
$$B=\tfrac{23}{5}.$$

To solve for A, we may use either of two methods.

Method 1. Let $x = 0$ in equation (2):

$$\begin{aligned} 3 &= -6A + 3B + 4C \\ &= -6A + 3\cdot\tfrac{23}{5} + 4\cdot\tfrac{33}{25}, \\ 0 &= -6A + 3\cdot\tfrac{18}{5} + 4\cdot\tfrac{33}{25}, \\ 6A &= 3\cdot\tfrac{134}{25} \quad \text{so} \quad A = \tfrac{67}{25}. \end{aligned}$$

Method 2. Differentiate equation (2) to give

$$8x + 2 = A\left[(x-2) + (x+3)\right] + B + 2C(x-2)$$

and then substitute $x = 2$ again:

$$\begin{aligned} 8\cdot 2 + 2 &= A(2+3) + B, \\ 18 &= 5A + B = 5A + \tfrac{23}{5}, \\ 5A &= 18 - \tfrac{23}{5} = \tfrac{67}{5}, \\ A &= \tfrac{67}{25}. \end{aligned}$$

This gives

$$\frac{4x^2+2x+3}{(x-2)^2(x+3)} = \frac{67}{25}\,\frac{1}{x-2} + \frac{23}{5}\,\frac{1}{(x-2)^2} + \frac{33}{25}\,\frac{1}{x+3}.$$

(At this point, it is a good idea to check your answer, either by adding up the right-hand side or by substituting a few values of x, using a calculator.)

We can now integrate:

$$\begin{aligned} \int \frac{4x^2+2x+3}{(x-2)^2(x+3)}\,dx &= \frac{67}{25}\int\frac{dx}{x-2} + \frac{23}{5}\int\frac{dx}{(x-2)^2} + \frac{33}{25}\int\frac{dx}{x+3} \\ &= \frac{67}{25}\ln|x-2| - \frac{23}{5}\,\frac{1}{x-2} + \frac{33}{25}\ln|x+3| + C. \end{aligned}$$

(b) Since the integrand "blows up" at $x = -3$ and $x = 2$, it only makes sense to evaluate definite integrals over intervals which do not contain these points; $[-1, 1]$ is such an interval. Thus, by (a), the definite integral is

$$\begin{aligned} &\left(\frac{67}{25}\ln|x-2| - \frac{23}{5}\,\frac{1}{x-2} + \frac{33}{25}\ln|x+3|\right)\Bigg|_{-1}^{1} \\ &\quad = \frac{67}{25}(\ln 1 - \ln 3) - \frac{23}{5}\left(\frac{1}{-1} - \frac{1}{-3}\right) + \frac{33}{25}(\ln 4 - \ln 2) \\ &\quad \cong -2.944 + 3.067 + 0.915 \cong 1.037. \ \blacktriangle \end{aligned}$$

Not every polynomial can be written as a product of linear factors. For instance, $x^2 + 1$ cannot be factored further (unless we use complex numbers) nor can any other quadratic function $ax^2 + bx + c$ for which $b^2 - 4ac < 0$; but any polynomial can, in principle, be factored into linear and quadratic factors. (This is proved in more advanced algebra texts.) This factorization is not always so easy to carry out in practice, but whenever we manage to factor the denominator of a rational function, we can integrate that function by the method of partial fractions.

Example 3 Integrate $\displaystyle\int \frac{1}{x^3-1}\,dx$.

Solution The denominator factors as $(x-1)(x^2+x+1)$, and x^2+x+1 cannot be

further factored (since $b^2 - 4ac = 1 - 4 = -3 < 0$). Now write

$$\frac{1}{x^3 - 1} = \frac{a}{x - 1} + \frac{Ax + B}{x^2 + x + 1}.$$

Thus $1 = a(x^2 + x + 1) + (x - 1)(Ax + B)$. We substitute values for x:

$$x = 1\colon 1 = 3a \qquad \text{so} \quad a = \tfrac{1}{3};$$

$$x = 0\colon 1 = \tfrac{1}{3} - B \qquad \text{so} \quad B = -\tfrac{2}{3}.$$

Comparing the x^2 terms, we get $0 = a + A$, so $A = -\frac{1}{3}$. Hence

$$\frac{1}{x^3 - 1} = \frac{1}{3}\left(\frac{1}{x - 1} - \frac{x + 2}{x^2 + x + 1}\right).$$

(This is a good point to check your work.)

Now

$$\int \frac{1}{x - 1}\,dx = \ln|x - 1| + C$$

and, writing $x + 2 = \frac{1}{2}(2x + 1) + \frac{3}{2}$,

$$\begin{aligned}\int \frac{x + 2}{x^2 + x + 1}\,dx &= \frac{1}{2}\int \frac{2x + 1}{x^2 + x + 1}\,dx + \frac{3}{2}\int \frac{dx}{(x + 1/2)^2 + 3/4} \\ &= \frac{1}{2}\ln|x^2 + x + 1| + \frac{3}{2}\cdot\sqrt{\frac{4}{3}}\,\tan^{-1}\left(\frac{x + 1/2}{\sqrt{3/4}}\right) + C \\ &= \frac{1}{2}\ln|x^2 + x + 1| + \sqrt{3}\,\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + C.\end{aligned}$$

Thus

$$\begin{aligned}\int \frac{dx}{x^3 - 1} &= \frac{1}{3}\ln|x - 1| - \frac{1}{6}\ln|x^2 + x + 1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + C \\ &= \frac{1}{3}\left[\frac{1}{2}\ln\left|\frac{(x - 1)^2}{x^2 + x + 1}\right| - \sqrt{3}\,\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right)\right] + C.\end{aligned}$$

Observe that the innocuous-looking integrand $1/(x^3 - 1)$ has brought forth both logarithmic and trigonometric functions. ▲

Now we are ready to set out a systematic method for the integration of $P(x)/Q(x)$ by partial fractions. (See the box on p. 469.) A few remarks may clarify the procedures given in the box. In case the denominator Q factors into n distinct linear factors, which we denote $Q = (x - r_1)(x - r_2)\ldots(x - r_n)$, we write

$$\frac{P}{Q} = \frac{\alpha_1}{x - r_1} + \frac{\alpha_2}{x - r_2} + \cdots + \frac{\alpha_n}{x - r_n}$$

and determine the n coefficients $\alpha_1, \ldots, \alpha_n$ by multiplying by Q and matching P to the resulting polynomial. The division in step 1 has guaranteed that P has degree at most $n - 1$, containing n coefficients. This is consistent with the number of constants $\alpha_1, \ldots, \alpha_n$ we have at our disposal. Similarly, if the denominator has repeated roots, or if there are quadratic factors in the denominator, it can be checked that the number of constants at our disposal is equal to the number of coefficients in the numerator to be matched. A system of n equations in n unknowns is likely to have a unique solution, and in this case, one can prove that it does.[2]

[2] See Review Exercise 88, Chapter 13 for a special case, or H. B. Fine, *College Algebra*, Dover, New York (1961), p. 241 for the general case.

Partial Fractions

To integrate $P(x)/Q(x)$, where P and Q are polynomials containing no common factor:

1. If the degree of P is larger than or equal to the degree of Q, divide Q into P by long division, obtaining a polynomial plus $R(x)/Q(x)$, where the degree of R is less than that of Q. Thus we need only investigate the case where the degree of P is less than that of Q.
2. Factor the denominator Q into linear and quadratic factors—that is, factors of the form $(x-r)$ and ax^2+bx+c. (Factor the quadratic expressions if $b^2-4ac>0$.)
3. If $(x-r)^m$ occurs in the factorization of Q, write down a sum of the form
$$\frac{a_1}{(x-r)}+\frac{a_2}{(x-r)^2}+\cdots+\frac{a_m}{(x-r)^m},$$
where $a_1, a_2, \ldots$ are constants. Do so for each factor of this form (using constants $b_1, b_2, \ldots, c_1, c_2, \ldots,$ and so on) and add the expressions you get. The constants $a_1, a_2, \ldots, b_1, b_2, \ldots,$ and so on will be determined in step 5.
4. If $(ax^2+bx+c)^p$ occurs in the factorization of Q with $b^2-4ac<0$, write down a sum of the form
$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_px+B_p}{(ax^2+bx+c)^p}.$$
Do so for each factor of this form and add the expressions you get. The constants $A_1, A_2, \ldots, B_1, B_2, \ldots$ are determined in step 5. Add this expression to the one obtained in step 3.
5. Equate the expression obtained in steps 3 and 4 to $P(x)/Q(x)$. Multiply through by $Q(x)$ to obtain an equation between two polynomials. Comparing coefficients of these polynomials, determine equations for the constants $a_1, a_2, \ldots, A_1, A_2, \ldots, B_1, B_2, \ldots$ and solve these equations. Sometimes the constants can be determined by substituting convenient values of x in the equality or by differentiation of the equality.
6. Check your work by adding up the partial fractions or substituting a few values of x.
7. Integrate the expression obtained in step 5 by using
$$\int \frac{dx}{(x-r)^j} = -\left[\frac{1}{(j-1)(x-r)^{j-1}}\right]+C, \qquad j>1$$
and $\displaystyle\int \frac{dx}{x-r} = \ln|x-r|+C.$

The terms with a quadratic denominator may be integrated by a manipulation which makes the derivative of the denominator appear in the numerator, together with completing the square (see Examples 3 and 6).

Example 4 Integrate $\int \frac{x^5 - x^4 + 1}{x^3 - x^2}\,dx$.

Solution First we divide out the fraction to get

$$\int \frac{x^5 - x^4 + 1}{x^3 - x^2}\,dx = \int \left(x^2 + \frac{1}{x^3 - x^2}\right)dx = \frac{1}{3}x^3 + \int \frac{dx}{x^3 - x^2}.$$

The denominator $x^3 - x^2$ is of degree 3, and the numerator is of degree zero. Thus we proceed to step 2 and factor:

$$x^3 - x^2 = x^2(x-1).$$

Here $x = x - 0$ occurs to the power 2, so by step 3, we write down

$$\frac{a_1}{x} + \frac{a_2}{x^2}.$$

We also add the term $b_1/(x-1)$ for the second factor.

$$\frac{a_1}{x} + \frac{a_2}{x^2} + \frac{b_1}{x-1}.$$

Since there are no quadratic factors, we omit step 4. By step 5, we equate the preceding expression to $1/(x^3 - x^2)$:

$$\frac{a_1}{x} + \frac{a_2}{x^2} + \frac{b_1}{x-1} = \frac{1}{x^2(x-1)}.$$

Then we multiply by $x^2(x-1)$:

$$a_1x(x-1) + a_2(x-1) + b_1x^2 = 1.$$

Setting $x = 0$, we get $a_2 = -1$. Setting $x = 1$, we get $b_1 = 1$. Comparing the coefficients of x^2 on both sides of the equation gives $a_1 + b_1 = 0$, so $a_1 = -b_1 = -1$. Thus $a_2 = -1$, $a_1 = -1$, and $b_1 = 1$. (We can check by substitution into the preceding equation: the left side is $(-1)x(x-1) - (x-1) + x^2$, which is just 1.)

Thus

$$\frac{1}{x^3 - x^2} = -\frac{1}{x^2} - \frac{1}{x} + \frac{1}{x-1},$$

and so

$$\int \frac{dx}{x^3 - x^2} = \frac{1}{x} - \ln|x| + \ln|x-1| + C = \frac{1}{x} + \ln\left|\frac{x-1}{x}\right| + C.$$

Finally,

$$\int \frac{x^5 - x^4 + 1}{x^3 - x^2}\,dx = \frac{1}{3}x^3 + \frac{1}{x} + \ln\left|\frac{x-1}{x}\right| + C. \blacktriangle$$

Example 5 Integrate $\int \frac{x^2}{(x^2 - 2)^2}\,dx$.

Solution The denominator factors as $(x - \sqrt{2})^2(x + \sqrt{2})^2$, so we write

$$\frac{x^2}{(x^2-2)^2} = \frac{a_1}{x - \sqrt{2}} + \frac{a_2}{(x-\sqrt{2})^2} + \frac{b_1}{x + \sqrt{2}} + \frac{b_2}{(x+\sqrt{2})^2}.$$

Thus

$$x^2 = a_1(x-\sqrt{2})(x+\sqrt{2})^2 + a_2(x+\sqrt{2})^2 + b_1(x+\sqrt{2})(x-\sqrt{2})^2 + b_2(x-\sqrt{2})^2.$$

We substitute values for x:

$$x=\sqrt{2} : 2 = 8a_2 \qquad \text{so} \quad a_2 = \tfrac{1}{4};$$

$$x=-\sqrt{2} : 2 = 8b_2 \qquad \text{so} \quad b_2 = \tfrac{1}{4}.$$

Therefore

$$\begin{aligned} x^2 &= a_1(x^2-2)(x+\sqrt{2}) + \tfrac{1}{4}(x^2+2\sqrt{2}x+2) \\ &\quad + b_1(x^2-2)(x-\sqrt{2}) + \tfrac{1}{4}(x^2-2\sqrt{2}\,x+2) \\ &= (a_1+b_1)x^3 + \left(\sqrt{2}\,a_1 + \tfrac{1}{2} - \sqrt{2}\,b_1\right)x^2 \\ &\quad + (-2a_1-2b_1)x - 2\sqrt{2}\,a_1 + 1 + 2\sqrt{2}\,b_1, \end{aligned}$$

and so

$$a_1 + b_1 = 0 \quad \text{and} \quad \sqrt{2}\,a_1 + \tfrac{1}{2} - \sqrt{2}\,b_1 = 1.$$

Thus

$$a_1 = \frac{1}{4\sqrt{2}}, \qquad b_1 = -\frac{1}{4\sqrt{2}}.$$

Hence

$$\begin{aligned} \frac{x^2}{(x^2-2)^2} &= \frac{1}{4\sqrt{2}\,(x-\sqrt{2})} + \frac{1}{4(x-\sqrt{2})^2} \\ &\quad - \frac{1}{4\sqrt{2}\,(x+\sqrt{2})} + \frac{1}{4(x+\sqrt{2})^2}, \end{aligned}$$

and so

$$\begin{aligned} \int \frac{x^2}{(x^2-2)^2}\,dx &= \frac{1}{4\sqrt{2}} \ln\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| - \frac{1}{4(x-\sqrt{2})} - \frac{1}{4(x+\sqrt{2})} + C \\ &= \frac{1}{4\sqrt{2}} \ln\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| - \frac{x}{2(x^2-2)} + C. \blacktriangle \end{aligned}$$

Example 6 Integrate $\displaystyle\int \frac{x^3}{(x-1)(x^2+2x+2)^2}\,dx.$

Solution For the factor $x-1$ we write

$$\frac{a_1}{x-1},$$

and for $(x^2+2x+2)^2$ (which does not factor further since x^2+2x+2 does not have real roots since $b^2-4ac = 4-4\cdot 1\cdot 2 = -4 < 0$) we write

$$\frac{A_1x+B_1}{x^2+2x+2} + \frac{A_2x+B_2}{(x^2+2x+2)^2}.$$

We then set

$$\frac{a_1}{x-1}+\frac{A_1x+B_1}{x^2+2x+2}+\frac{A_2x+B_2}{(x^2+2x+2)^2}=\frac{x^3}{(x-1)(x^2+2x+2)^2}$$

and multiply by $(x-1)(x^2+2x+2)^2$:

$$a_1(x^2+2x+2)^2+(A_1x+B_1)(x-1)(x^2+2x+2)$$
$$+(A_2x+B_2)(x-1)=x^3.$$

Setting $x=1$ gives $a_1(25)=1$ or $a_1=\frac{1}{25}$. Expanding the left-hand side, we get:

$$\tfrac{1}{25}(x^4+4x^3+8x^2+8x+4)+A_1x^4+(A_1+B_1)x^3+B_1x^2$$
$$-2A_1x-2B_1+A_2x^2+(B_2-A_2)x-B_2=x^3.$$

Comparing coefficients:

$$x^4:\ \tfrac{1}{25}+A_1=0; \tag{3}$$
$$x^3:\ \tfrac{4}{25}+(A_1+B_1)=1; \tag{4}$$
$$x^2:\ \tfrac{8}{25}+B_1+A_2=0; \tag{5}$$
$$x:\ \tfrac{8}{25}-2A_1+(B_2-A_2)=0; \tag{6}$$
$$x^0(=1):\ \tfrac{4}{25}-2B_1-B_2=0. \tag{7}$$

Thus

$A_1=-\frac{1}{25}$ (from equation (3));
$B_1=\frac{22}{25}$ (from equation (4));
$A_2=-\frac{30}{25}$ (from equation (5));
$B_2=-\frac{40}{25}$ (from equation (6)).

At this stage you may check the algebra by substitution into equation (7). Algebraic errors are easy to make in integration by partial fractions.

We have thus far established

$$\frac{x^3}{(x-1)(x^2+2x+2)^2}=\frac{1}{25}\left[\frac{1}{x-1}+\frac{-x+22}{x^2+2x+2}+\frac{-30x-40}{(x^2+2x+2)^2}\right].$$

We compute the integrals of the first two terms as follows:

$$\int\frac{1}{x-1}\,dx=\ln|x-1|+C$$

$$\int\frac{-x+22}{x^2+2x+2}\,dx=\int\frac{-x-1+23}{x^2+2x+2}\,dx$$
$$=-\frac{1}{2}\int\frac{2x+2}{x^2+2x+2}\,dx+23\int\frac{dx}{x^2+2x+2}$$
$$=-\frac{1}{2}\ln|x^2+2x+2|+23\int\frac{dx}{(x+1)^2+1}$$
$$=-\frac{1}{2}\ln|x^2+2x+2|+23\tan^{-1}(x+1)+C.$$

Finally, for the last term, we rearrange the numerator to make the derivative of the quadratic polynomial in the denominator appear:

$$\int \frac{-30x-40}{(x^2+2x+2)^2}\,dx = \int \frac{-15(2x+2)-10}{(x^2+2x+2)^2}\,dx$$

$$= 15\cdot\frac{1}{(x^2+2x+2)} - 10\int \frac{1}{\left[(x+1)^2+1\right]^2}\,dx.$$

Let $x+1=\tan\theta$, so $dx=\sec^2\theta\,d\theta$ and $(x+1)^2+1=\sec^2\theta$. Then

$$\int \frac{1}{\left[(x+1)^2+1\right]^2}\,dx = \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta} = \int \cos^2\theta\,d\theta$$

$$= \int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$$

$$= \frac{1}{2}\tan^{-1}(x+1) + \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{1}{2}\tan^{-1}(x+1) + \frac{1}{2}\cdot\frac{x+1}{(x+1)^2+1} + C$$

(see Fig. 10.2.1).

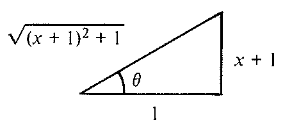


Figure 10.2.1. Geometry of the substitution $x+1=\tan\theta$.

Adding the results obtained above, we find

$$\int \frac{x^3}{(x-1)(x^2+2x+2)}\,dx$$

$$= \frac{1}{25}\left[\ln|x-1| - \frac{1}{2}\ln(x^2+2x+2) + 23\tan^{-1}(x+1)\right.$$

$$\left. + 15\frac{1}{x^2+2x+2} - 5\tan^{-1}(x+1) - 5\frac{x+1}{x^2+2x+2}\right] + C$$

$$= \frac{1}{25}\left[\ln\left(\frac{|x-1|}{\sqrt{x^2+2x+2}}\right) + 18\tan^{-1}(x+1) + \frac{10-5x}{x^2+2x+2}\right] + C. \blacktriangle$$

Integrands with a single power $(x-a)^r$ in the denominator may appear to require partial fractions but are actually easiest to evaluate using a simple substitution.

Example 7 Integrate $\int \frac{x^3+2x+1}{(x-1)^5}\,dx$.

Solution Let $u=x-1$ so $du=dx$ and $x=u+1$. Then

$$\int \frac{x^3+2x+1}{(x-1)^5}\,dx = \int \frac{(u+1)^3+2(u+1)+1}{u^5}\,du$$

$$= \int \frac{u^3+3u^2+5u+4}{u^5}\,du = \int\left(\frac{1}{u^2}+\frac{3}{u^3}+\frac{5}{u^4}+\frac{4}{u^5}\right)du$$

$$= -\frac{1}{u} - \frac{3}{2u^2} - \frac{5}{3u^3} - \frac{4}{4u^4} + C$$

$$= -\left[\frac{1}{x-1} + \frac{3}{2(x-1)^2} + \frac{5}{3(x-1)^3} + \frac{1}{(x-1)^4}\right] + C. \blacktriangle$$

To conclude this section, we present a couple of techniques in which an integrand is converted by a substitution into a rational function which can then be integrated by partial fractions. The first such technique, called the method of *rationalizing substitutions*, applies when an integrand involves a fractional power. The idea is to express the fractional power as an integer power of a new variable.

Example 8 Eliminate the fractional power from $\int \frac{(1+x)^{2/3}}{1+2x}\,dx$.

Solution To get rid of the fractional power, substitute $u=(1+x)^{1/3}$. Then $u^3=1+x$ and $3u^2\,du=dx$, so the integral becomes

$$\int \frac{u^2}{1+2(u^3-1)}\cdot 3u^2\,du=\int \frac{3u^4\,du}{2u^3-1}. \blacktriangle$$

After the rationalizing substitution as made, the method of partial functions can be used to evaluate the integral. (Evaluating the integral above is left as Exercise 24).

Example 9 Try the substitution $u=\sqrt[3]{x^2+4}$ in the integrals:

(a) $\int \frac{x^4\,dx}{\sqrt[3]{x^2+4}}$ and (b) $\int \frac{2x^7\,dx}{\sqrt[3]{x^2+4}}$.

Solution We have $u^3=x^2+4$ and $3u^2\,du=2x\,dx$, so integral (a) becomes

$$\int \frac{x^4}{u}\cdot\frac{3u^2}{2x}\,du=\int \frac{3}{2}ux^3\,du,$$

from which we cannot eliminate x without introducing a new fractional power. However, (b) is

$$\int \frac{2x^7}{u}\cdot\frac{3u^2}{2x}\,du=\int 3ux^6\,du=\int 3u(u^3-4)^3\,du$$

(which can be evaluated as the integral of a polynomial). The reason the method works in case (b) lies in the special relation between the exponents of x inside and outside the radical (see Exercise 27). $\blacktriangle$

The second general technique applies when the integrand is built up by rational operations from $\sin x$ and $\cos x$ (and hence from the other trigonometric functions as well). The substitution $u=\tan(x/2)$ turns such an integrand into a rational function of u by virtue of the following trigonometric identities:

$$\sin x=\frac{2u}{1+u^2}, \tag{8}$$

$$\cos x=\frac{1-u^2}{1+u^2}, \tag{9}$$

and

$$dx=\frac{2\,du}{1+u^2}. \tag{10}$$

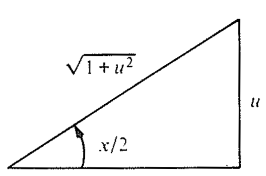


Figure 10.2.2. With the substitution $\tan(x/2)=u$, $\sin(x/2)=u/\sqrt{1+u^2}$.

To prove equation (8), use the addition formula

$$\sin x=\sin\left(\frac{x}{2}+\frac{x}{2}\right)=2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$=2\left(\frac{u}{\sqrt{1+u^2}}\right)\left(\frac{1}{\sqrt{1+u^2}}\right)=\frac{2u}{1+u^2} \quad \text{(see Fig. 10.2.2).}$$

Similarly we derive equation (9). Equation (10) holds since

$$\frac{du}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}\left[\tan^2\left(\frac{x}{2}\right) + 1\right] = \frac{1}{2}(1 + u^2)$$

Example 10 Evaluate $\displaystyle\int \frac{dx}{2 + \cos x}$.

Solution Using equations (8), (9), and (10), we convert the integral to

$$\int \frac{2\,du}{1 + u^2} \cdot \frac{1}{2 + \left[(1 - u^2)/(1 + u^2)\right]} = \int \frac{2\,du}{2 + 2u^2 + 1 - u^2} = \int \frac{2\,du}{3 + u^2},$$

which is rational in u. No partial fraction decomposition is necessary; the substitution $u = \sqrt{3}\tan\theta$ converts the integral to

$$\int \frac{2 \cdot \sqrt{3}\sec^2\theta\, d\theta}{3 + 3\tan^2\theta} = \frac{2}{\sqrt{3}}\int d\theta = \frac{2\theta}{\sqrt{3}} + C$$

(using the identity $1 + \tan^2\theta = \sec^2\theta$). Writing the answer in terms of x, we get

$$\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) + C$$

for our final answer. ▲

> **Rational Expressions in $\sin x$ and $\cos x$**
>
> If $f(x)$ is a rational expression in $\sin x$ and $\cos x$, then substitute $u = \tan(x/2)$. Using equations (8), (9), and (10), transform $\int f(x)\,dx$ into the integral of a rational function of u to which the method of partial fractions can be applied.

Example 11 Find $\displaystyle\int_0^{\pi/4} \sec\theta\, d\theta$.

Solution First we find $\int \sec\theta\, d\theta = \int [d\theta/\cos\theta]$. We use equations (9) and (10) (with x replaced by θ) to get

$$\begin{aligned}\int \frac{d\theta}{\cos\theta} &= \int \frac{1 + u^2}{1 - u^2}\,\frac{2\,du}{1 + u^2} = 2\int \frac{du}{1 - u^2}\\ &= \int\left(\frac{1}{1 + u} + \frac{1}{1 - u}\right)du = \ln|1 + u| - \ln|1 - u| + C\\ &= \ln\left|\frac{1 + u}{1 - u}\right| + C = \ln\left|\frac{1 + \tan(\theta/2)}{1 - \tan(\theta/2)}\right| + C.\end{aligned}$$

(Compare this procedure with the method we used to find $\int \sec\theta\, d\theta$ in Example 6(b), Section 10.1.) Finally,

$$\begin{aligned}\int_0^{\pi/4} \sec\theta\, d\theta &= \left(\ln\left|\frac{1 + \tan(\theta/2)}{1 - \tan(\theta/2)}\right|\right)\Bigg|_0^{\pi/4} = \ln\left(\frac{1 + \tan(\pi/8)}{1 - \tan(\pi/8)}\right) - \ln 1\\ &= \ln\left(\frac{1 + \tan(\pi/8)}{1 - \tan(\pi/8)}\right) \approx 0.881. \ ▲\end{aligned}$$

Exercises for Section 10.2

Evaluate the integrals in Exercises 1–12 using the method of partial fractions.

1. $\displaystyle\int \frac{1}{(x-2)^2(x^2+1)^2}\,dx.$
2. $\displaystyle\int \frac{x^4+2x^3+3}{(x-4)^6}\,dx.$
3. $\displaystyle\int_0^1 \frac{x^4}{(x^2+1)^2}\,dx$
4. $\displaystyle\int_0^1 \frac{2x^3-1}{x^2+1}\,dx$
5. $\displaystyle\int \frac{x^2}{(x-2)(x^2+2x+2)}\,dx.$
6. $\displaystyle\int \frac{dx}{(x-2)(x^2+3x+1)}$
7. $\displaystyle\int_2^4 \frac{x^3+1}{x^3-1}\,dx$
8. $\displaystyle\int_0^1 \frac{dx}{8x^3+1}$
9. $\displaystyle\int \frac{x}{x^4+2x^2-3}\,dx.$
10. $\displaystyle\int \frac{2x^2-x+2}{x^5+2x^3+x}\,dx$
11. $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos x\,dx}{\sin x+\sin^3 x}$
12. $\displaystyle\int_0^{\pi/4} \frac{(\sec^2 x+1)\sec^2 x\,dx}{1+\tan^3 x}$

Evaluate the integrals in Exercises 13–16 using a rationalizing substitution.

13. $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx$
14. $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$
15. $\displaystyle\int x\sqrt[3]{x^2+1}\,dx$
16. $\displaystyle\int x^3\sqrt[3]{x^2+1}\,dx$

Evaluate the integrals in Exercises 17–20.

17. $\displaystyle\int \frac{dx}{1+\sin x}$
18. $\displaystyle\int \frac{dx}{1+2\cos x}$
19. $\displaystyle\int_0^{\pi/4} \frac{d\theta}{1+\tan\theta}$
20. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{d\theta}{1-\cos\theta}$
21. Find the volume of the solid obtained by revolving the region under the graph of the function $y = 1/[(1-x)(1-2x)]$ on $[5,6]$ about the y axis.
22. Find the center of mass of the region under the graph of $1/(x^2+4)$ on $[1,3]$.
23. Evaluate $\displaystyle\int \frac{(1+x)^{3/2}}{x}\,dx.$

★24. Evaluate the integral in Example 8.

25. A chemical reaction problem leads to the following equation:
$$\int \frac{dx}{(80-x)(60-x)} = k\int dt, \qquad k = \text{constant}.$$
In this formula, $x(t)$ is the number of kilograms of reaction product present after t minutes, starting with 80 kilograms and 60 kilograms of two reacting substances which obey the *law of mass action*.
 (a) Integrate to get a logarithmic formula involving x and t ($x = 0$ when $t = 0$).
 (b) Convert the answer to an exponential formula for x (assume $x < 60$).
 (c) How much reaction product is present after 15 minutes, assuming $x = 20$ when $t = 10$?

26. Partial fractions appear in electrical engineering as a convenient means of analyzing and describing circuit responses to applied voltages. By means of the Laplace transform, circuit responses are associated with rational funcitons. Partial fraction methods are used to decompose these rational functions to elementary quotients, which are recognizable to engineers as arising from standard kinds of circuit responses. For example, from
$$\frac{s+1}{(s+2)(s^2+1)(s^2+4)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+1} + \frac{Ds+2E}{s^2+4},$$
an engineer can easily see that this rational function represents the response
$$Ae^{-2t} + B\cos t + C\sin t + D\cos 2t + E\sin 2t.$$
Find the constants A, B, C, D, E.

★27. (a) Try evaluating $\int (x^m+b)^{p/q}x^r\,dx$, where m, p, q, and r are integers and b is a constant by the substitution $u = (x^m+b)^{1/q}$.
 (b) Show that the integral in (a) becomes the integral of a rational function of u when the number $r - m + 1$ is evenly divisible by m.

★28. Any rational function which has the form $p(x)/(x-a)^m(x-b)^n$, where $\deg p < m+n$, can be integrated in the following way:
 (i) Write
$$\frac{p(x)}{(x-a)^m(x-b)^n} = \frac{q(x)}{(x-a)^m} + \frac{r(x)}{(x-b)^n},$$
 where $\deg q < m$ and $\deg r < n$.
 (ii) Integrate each term, using the substitutions $u = x - a$ and $v = x - b$.
 (a) Use this procedure to find
$$\int \frac{dx}{(x-2)^2(x-3)^3}.$$
 (b) Find the same integral by the ordinary partial fraction method.
 (c) Compare answers and the efficiency of the two methods.

10.3 Arc Length and Surface Area

Integration can be used to find the length of graphs in the plane and the area of surfaces of revolution.

In Sections 4.6, 9.1, and 9.2, we developed formulas for areas under and between graphs and for volumes of solids of revolution. In this section we continue applying integration to geometry and obtain formulas for lengths and areas.

The length of a piece of curve in the plane is sometimes called the *arc length* of the curve. As we did with areas and volumes, we assume that the length exists and will try to express it as an integral. For now, we confine our attention to curves which are graphs of functions; general curves are considered in the next section.

We shall begin with an argument involving infinitesimals to derive the formula for arc length. Following this, a different derivation will be given using step functions. The second method is the "honest" one, but it is also more technical.

We consider a curve that is a graph $y = f(x)$ from $x = a$ to $x = b$, as in Fig. 10.3.1. The curve may be thought of as being composed of infinitely

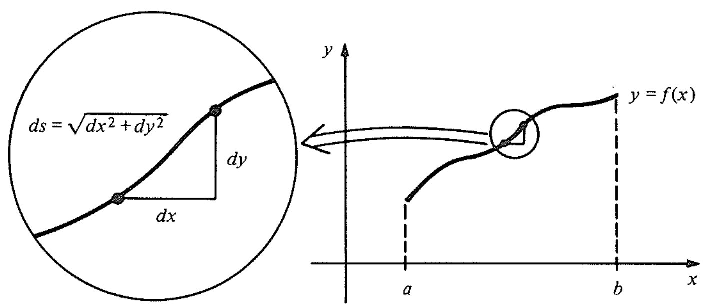


Figure 10.3.1. An "infinitesimal segment" of the graph of f.

many infinitesimally short segments. By the theorem of Pythagoras, the length ds of each segment is equal to $\sqrt{dx^2 + dy^2}$. But $dy/dx = f'(x)$, so $dy = f'(x)\,dx$ and $ds = \sqrt{dx^2 + [f'(x)]^2 dx^2} = \sqrt{1 + [f'(x)]^2}\, dx$. To get the total length, we add up all the infinitesimal lengths: $\int_a^b ds = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$.

Length of Curves

Suppose that the function f is continuous on $[a, b]$, and that the derivative f' exists and is continuous (except possibly at finitely many points) on $[a, b]$. Then the length of the graph of f on $[a, b]$ is:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \tag{1}$$

Let us check that formula (1) gives the right result for the length of an arc of a circle.

Example 1 Use integration to find the length of the graph of $f(x)=\sqrt{1-x^2}$ on $[0,b]$, where $0<b<1$. Then find the length geometrically and compare the results.

Solution By formula (1), the length is $\int_0^b\sqrt{1+[f'(x)]^2}\,dx$, where $f(x)=\sqrt{1-x^2}$. We have

$$f'(x)=\frac{-x}{\sqrt{1-x^2}},\qquad [f'(x)]^2=\frac{x^2}{1-x^2},\qquad 1+[f'(x)]^2=\frac{1}{1-x^2}.$$

Hence

$$L=\int_0^b\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}(b)-\sin^{-1}(0)=\sin^{-1}(b).$$

Examining Fig. 10.3.2, we see that $\sin^{-1}(b)$ is equal to θ, the angle intercepted

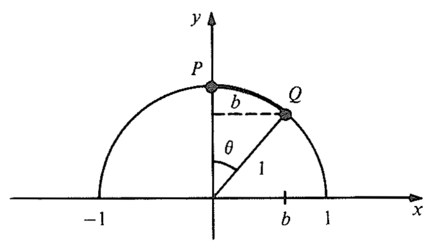


Figure 10.3.2. The length of the arc PQ is $\theta=\sin^{-1}b$.

by the arc whose length we are computing. By the definition of radian measure, the length of the arc is equal to the angle $\theta=\sin^{-1}(b)$, which agrees with our calculation by means of the integral. ▲

Example 2 Find the length of the graph of $f(x)=(x-1)^{3/2}+2$ on $[1,2]$.

Solution We are given $f(x)=(x-1)^{3/2}+2$ on $[1,2]$. Since $f'(x)=\frac{3}{2}(x-1)^{1/2}$, the length of the graph is

$$\begin{aligned}\int_a^b\sqrt{1+[f'(x)]^2}\,dx&=\int_1^2\sqrt{1+\frac{9}{4}(x-1)}\,dx=\frac{1}{2}\int_1^2\sqrt{9x-5}\,dx\\&=\frac{1}{18}\int_4^{13}u^{1/2}\,du\qquad(\text{where } u=9x-5)\\&=\tfrac{1}{27}(13^{3/2}-8)\approx 1.44.\ ▲\end{aligned}$$

Due to the square root, the integral in formula (1) is often difficult or even impossible to evaluate by elementary means. Of course, we can always approximate the result numerically (see Section 11.5 for specific examples). The following example shows how a simple-looking function can lead to a complicated integral for arc length.

Example 3 Find the length of the parabola $y=x^2$ from $x=0$ to $x=1$.

Solution We substitute $f(x)=x^2$ and $f'(x)=2x$ into formula (1):

$$L=\int_0^1\sqrt{1+(2x)^2}\,dx=2\int_0^1\sqrt{(\tfrac{1}{2})^2+x^2}\,dx.$$

Now substitute $x = \frac{1}{2}\tan\theta$ and $\sqrt{(1/2)^2 + x^2} = \frac{1}{2}\sec\theta$:

$$\int \sqrt{\left(\tfrac{1}{2}\right)^2 + x^2}\, dx = \int \left(\frac{1}{2}\sec\theta\right)\left(\frac{1}{2}\sec^2\theta\, d\theta\right) = \frac{1}{4}\int \sec^3\theta\, d\theta.$$

We evaluate the integral of $\sec^3\theta$ using the following trickery[3]:

$$\begin{aligned}\int \sec^3\theta\, d\theta &= \int \sec\theta \sec^2\theta\, d\theta = \int \sec\theta(\tan^2\theta + 1)\, d\theta\\ &= \int \sec\theta\tan^2\theta\, d\theta + \int \sec\theta\, d\theta\\ &= \int (\sec\theta\tan\theta)\tan\theta\, d\theta + \ln|\sec\theta + \tan\theta|.\end{aligned}$$

(see Example 10, Section 10.2.) Now integrate by parts:

$$\begin{aligned}\int (\sec\theta\tan\theta)\tan\theta\, d\theta &= \int \frac{d}{d\theta}(\sec\theta)\tan\theta\, d\theta\\ &= \sec\theta\tan\theta - \int \sec\theta\sec^2\theta\, d\theta\\ &= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta.\end{aligned}$$

Substituting this formula into the last expression for $\int\sec^3\theta\, d\theta$ gives

$$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \ln|\sec\theta + \tan\theta|;$$

so

$$\int \sec^3\theta\, d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) + C.$$

Since $2x = \tan\theta$ and $\sec\theta = 2\cdot\sqrt{(1/2)^2 + x^2} = \sqrt{1 + 4x^2}$, we can express the integral $\int\sec^3\theta\, d\theta$ in terms of x as

$$x\sqrt{1 + 4x^2} + \tfrac{1}{2}\ln|2x + \sqrt{1 + 4x^2}\,| + C.$$

[One may also evaluate the integral $\int\sqrt{\left(\frac{1}{2}\right)^2 + x^2}\, dx$ using integral formula (43) from the endpapers.] Substitution into the formula for L gives

$$\begin{aligned}L &= \tfrac{1}{2}\left(x\sqrt{1 + 4x^2} + \tfrac{1}{2}\ln|2x + \sqrt{1 + 4x^2}\,|\right)\Big|_0^1\\ &= \tfrac{1}{2}\left[\sqrt{5} + \tfrac{1}{2}\ln(2 + \sqrt{5})\right] \approx 1.479. \ \blacktriangle\end{aligned}$$

Example 4 Express the length of the graph of $f(x) = \sqrt{1 - k^2x^2}$ on $[0, b]$ as an integral.

Solution We get

$$f'(x) = -\frac{k^2x}{\sqrt{1 - k^2x^2}}, \qquad \text{so} \qquad \sqrt{1 + [f'(x)]^2} = \sqrt{\frac{1 + (k^4 - k^2)x^2}{1 - k^2x^2}},$$

[3] We can also write

$$\int \sec^3\theta\, d\theta = \int \frac{\cos\theta}{\cos^4\theta}\, d\theta = \int \frac{\cos\theta}{(1 - \sin^2\theta)^2}\, d\theta = \int \frac{du}{(1 - u^2)^2} \qquad (u = \sin\theta).$$

The last integral may now be evaluated by partial fractions.

Thus,

$$L = \int_0^b \sqrt{\frac{1 + (k^4 - k^2)x^2}{1 - k^2x^2}}\, dx.$$

It turns out that the antiderivative

$$\int \sqrt{\frac{1 + (k^4 - k^2)x^2}{1 - k^2x^2}}\, dx$$

cannot be expressed (unless $k^2 = 0$ or 1) in terms of algebraic, trigonometric, or exponential functions. It is a new kind of function called an *elliptic* function. (See Review Exercises 85 and 92 for more examples of such functions.) ▲

We now turn to the derivation of formula (1) using step functions.

Our first principle for arc length is that the length of a straight line segment is equal to the distance between its endpoints. Thus, if $f(x) = mx + q$ on $[a,b]$, the endpoints (see Fig. 10.3.3) of the graph are $(a, ma + q)$ and

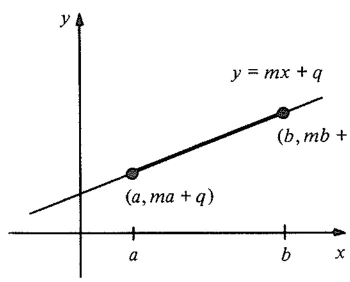


Figure 10.3.3. The length of the dark segment is $(b - a)\sqrt{1 + m^2}$, where m is the slope.

$(b, mb + q)$, and the distance between them is

$$\sqrt{(a-b)^2 + \left[(ma+q) - (mb+q)\right]^2} = \sqrt{(a-b)^2 + m^2(a-b)^2}$$
$$= (b-a)\sqrt{1 + m^2}\,.$$

(Since $a < b$, the square root of $(a - b)^2$ is $b - a$.)

Our strategy, as in Chapter 9, will be to interpret the arc length for a simple curve as an integral and then use the same formula for general curves. In the case of the straight line segment, $f(x) = mx + q$, whose length between $x = a$ and $x = b$ is $(b - a)\sqrt{1 + m^2}$, we can interpret m as the derivative $f'(x)$, so that

$$\text{Length} = (b - a)\sqrt{1 + m^2} = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx.$$

Since the formula for the length is an integral of f', rather than of f, it is natural to look next at the functions for which f' is a step function. If f' is constant on an interval, f is linear on that interval; thus the functions with which we will be dealing are the *piecewise linear* (also called *ramp*, or *polygonal*) functions.

To obtain a piecewise linear function, we choose a partition of the interval $[a,b]$, say, $(x_0, x_1, \ldots, x_n)$ and specify the values $(y_0, y_1, \ldots, y_n)$ of

the function f at these points. For each $i = 1, 2, \ldots, n$, we then connect the point (x_{i-1}, y_{i-1}) to the point (x_i, y_i) by a straight line segment (see Fig. 10.3.4).

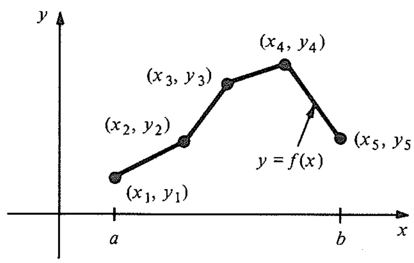


Figure 10.3.4. The graph of a piecewise linear function.

The function $f(x)$ is differentiable on each of the intervals (x_{i-1}, x_i), where its derivative is constant and equal to the slope $(y_i - y_{i-1})/(x_i - x_{i-1})$. Thus the function $\sqrt{1 + [f'(x)]^2}$ is a step function $[a, b]$, with value

$$k_i = \sqrt{1 + \left(\frac{y_i - y_{i-1}}{x_i - x_{i-1}} \right)^2}$$

on (x_{i-1}, x_i).[4] Therefore,

$$\begin{aligned} \int_a^b \sqrt{1 + [f'(x)]^2}\, dx &= \sum_{i=1}^{n} k_i \Delta x_i \\ &= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{y_i - y_{i-1}}{x_i - x_{i-1}} \right)^2}\, (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}\, . \end{aligned}$$

Note that the ith term in this sum, $\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$, is just the length of the segment of the graph of f between (x_{i-1}, y_{i-1}) and (x_i, y_i).

Now we invoke a second principle of arc length: if n curves are placed end to end, the length of the total curve is the sum of the lengths of the pieces. Using this principle, we see that the preceding sum is just the length of the graph of f on $[a, b]$. So we have now shown, for piecewise linear functions, that the length of the graph of f on $[a, b]$ equals the integral $\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$.

Example 5 Let the graph of f consist of straight line segments joining $(1, 0)$ to $(2, 1)$ to $(3, 3)$ to $(4, 1)$. Verify that the length of the graph, as computed directly, is given by the formula $\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$.

Solution The graph is sketched in Fig. 10.3.5. The length is

$$d_1 + d_2 + d_{\_} = \sqrt{1 + 1^2} + \sqrt{1 + 2^2} + \sqrt{1 + (-2)^2} = \sqrt{2} + 2\sqrt{5}\, .$$

[4] Actually, $\sqrt{1 + [f'(x)]^2}$ is not defined at the points $x_0, x_1, \ldots, x_n$, but this does not matter when we take its integral, since the integral is not affected by changing the value of the integrand at isolated points.

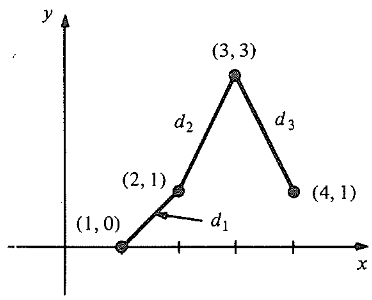


Figure 10.3.5. The length of this graph is $d_1 + d_2 + d_3$.

On the other hand,

$$f'(x) = \begin{cases} 1 & \text{on } (1,2) \\ 2 & \text{on } (2,3) \\ -2 & \text{on } (3,4) \end{cases}$$

(and is not defined at $x = 1, 2, 3, 4$). Thus, by the definition of the integral of a step function (see Section 4.3),

$$\int_1^4 \sqrt{1 + [f'(x)]^2}\, dx = (\sqrt{1+1^2}) \cdot 1 + (\sqrt{1+2^2}) \cdot 1 + \left[\sqrt{1+(-2)^2}\,\right] \cdot 1$$
$$= \sqrt{2} + 2\sqrt{5}$$

which agrees with the preceding answer. ▲

Justifying the passage from step functions to general functions is more complicated than in the case of area, since we cannot, in any straightforward way, squeeze a general curve between polygons as far as length is concerned. Nevertheless, it is plausible that any reasonable graph can be approximated by a piecewise linear function, so formula (1) should carry over. These considerations lead to the technical conditions stated in conjunction with formula (1).

If we revolve the region R under the graph of $f(x)$ (assumed nonnegative) on $[a, b]$, about the x axis, we obtain a solid of revolution S. In Section 9.1 we saw how to express the volume of such a solid as an integral. Suppose now that instead of revolving the region, we revolve the graph $y = f(x)$ itself. We obtain a curved surface Σ, called a *surface of revolution*, which forms part of the boundary of S. (The remainder of the boundary consists of the disks at the ends of the solid, which have radii $f(a)$ and $f(b)$; see Fig. 10.3.6.) Our next goal is to obtain a formula for the *area* of the surface Σ. Again we give the argument using infinitesimals first.

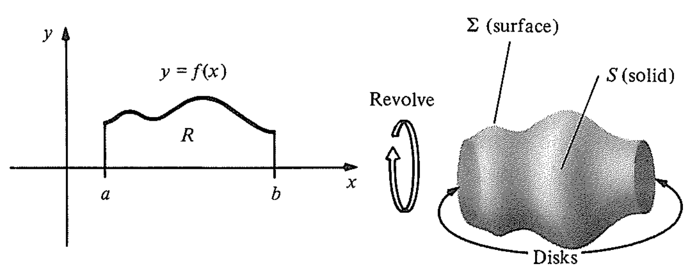


Figure 10.3.6. The boundary of the solid of revolution S consists of the surface of revolution Σ obtained by revolving the graph, together with two disks.

Referring to Figure 10.3.7, we may think of a smooth surface of revolution as being composed of infinitely many infinitesimal bands, as in Fig.

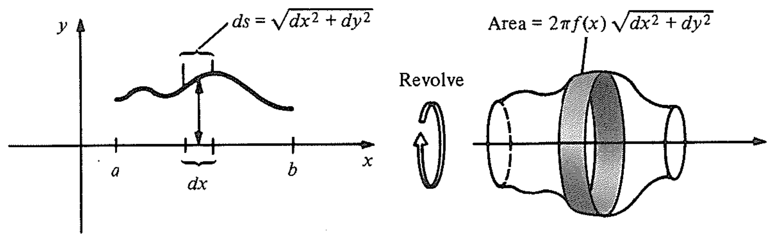


Figure 10.3.7. The surface of revolution may be considered as composed of infinitely many infinitesimal frustums.

10.3.7. The area of each band is equal to its circumference $2\pi f(x)$ times its width $ds = \sqrt{dx^2 + dy^2}$, so the total area is

$$\int_a^b 2\pi f(x)\sqrt{dx^2 + dy^2} = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

since $dy = f'(x)\,dx$.

Area of a Surface of Revolution about the x Axis

The area of the surface obtained by revolving the graph of $f(x)$ ($\geqslant 0$) on $[a,b]$ about the x axis is

$$A = 2\pi\int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx = 2\pi\int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \quad (2)$$

We now check that formula (2) gives the correct area for a sphere.

Example 6 Find the area of the spherical surface of radius r obtained by revolving the graph of $y = \sqrt{r^2 - x^2}$ on $[-r,r]$ about the x axis.

Solution As in Example 1, we have $\sqrt{1 + [f'(x)]^2} = r/\sqrt{r^2 - x^2}$, so the area is

$$\int_{-r}^{r} 2\pi\sqrt{r^2 - x^2}\,\frac{r}{\sqrt{r^2 - x^2}}\, dx = 2\pi r\int_{-r}^{r} dx = 2\pi r \cdot 2r = 4\pi r^2$$

which is the usual value for the area of a sphere. ▲

If, instead of the entire sphere, we take the band obtained by restricting x to $[a,b]$ $(-r \leqslant a \leqslant b \leqslant r)$, the area is $2\pi\int_a^b r\,dx = 2\pi r(b-a)$. Thus the area obtained by slicing a sphere by two parallel planes and taking the middle piece is equal to $2\pi r$ times the distance between the planes, regardless of where the two planes are located (see Fig. 10.3.8). Why doesn't the "longer" band around the middle have more area?

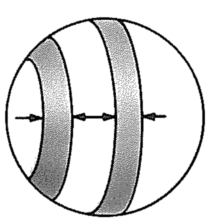


Figure 10.3.8. Bands of equal width have equal area.

As with arc length, the factor $\sqrt{1 + [f'(x)]^2}$ in the integrand sometimes makes it impossible to evaluate the surface area integrals by any means other than numerical methods (see Section 11.5). To get a problem which can be solved, we must choose f carefully.

Example 7 Find the area of the surface obtained by revolving the graph of x^3 on $[0, 1]$ about the x axis.

Solution We find that $f'(x) = 3x^2$ and $\sqrt{1 + [f'(x)]^2} = \sqrt{1 + 9x^4}$, so

$$A = 2\pi \int_0^1 \sqrt{1 + 9x^4}\, x^3\, dx = \frac{\pi}{2} \int_0^1 \sqrt{1 + 9u}\, du \qquad (u = x^4,\ du = 4x^3\, dx)$$

$$= \frac{\pi}{18} \int_0^9 (1 + v)^{1/2} dv \qquad \left(u = \frac{1}{9} v,\ du = \frac{1}{9}\, dv\right)$$

$$= \frac{\pi}{18} \left[\frac{2}{3} (1 + v)^{3/2} \right]\Bigg|_0^9 = \frac{\pi}{27} (10^{3/2} - 1) \approx 3.56. \ \blacktriangle$$

By a method similar to that for deriving equation (2), we can derive a formula for the area obtained by revolving the graph $y = f(x)$ about the y axis for $a \leqslant x \leqslant b$. Referring to Figure 10.3.9, the area of the shaded band is

$$\text{Width} \times \text{Circumference} = ds \cdot 2\pi x = 2\pi x \sqrt{dx^2 + dy^2}$$

Thus the surface area is $\int_a^b 2\pi x \sqrt{1 + [f'(x)]^2}\, dx$.

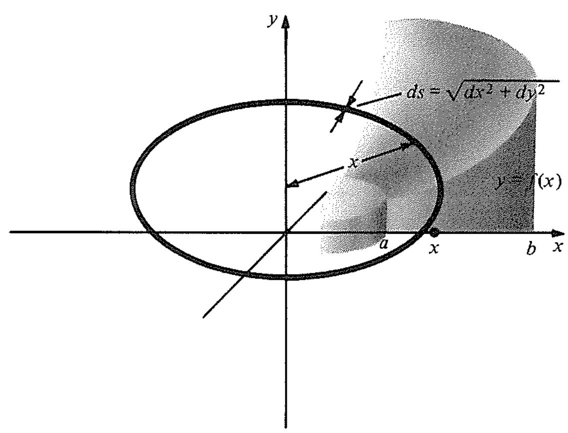


Figure 10.3.9. Rotating $y = f(x)$ about the y axis.

Area of a Surface of Revolution about the y Axis

The area of the surface obtained by revolving the graph of $f(x)$ ($\geqslant 0$) on $[a, b]$ about the y axis is

$$A = 2\pi \int_a^b x \sqrt{1 + [f'(x)]^2}\, dx. \tag{3}$$

Example 8 Find the area of the surface obtained by revolving the graph $y = x^2$ about the y axis for $1 \leqslant x \leqslant 2$.

Solution If $f(x) = x^2$, $f'(x) = 2x$ and $\sqrt{1 + [f'(x)]^2} = \sqrt{1 + 4x^2}$. Then

$$A = 2\pi \int_1^2 x\sqrt{1 + 4x^2}\, dx = \frac{\pi}{4} \int_5^{17} u^{1/2} du \qquad (u = 1 + 4x^2,\ du = 8x\, dx)$$

$$= \frac{\pi}{4} \left(\frac{2}{3} u^{3/2} \right)\Bigg|_5^{17} = \frac{\pi}{6} (17^{3/2} - 5^{3/2}) \approx 30.85. \ \blacktriangle$$

Finally we sketch how one derives formula (2) using step functions. The derivation of formula (3) is similar (see Exercise 41).

If $f(x) = k$, a constant, the surface is a cylinder of radius k and height $b - a$. Unrolling the cylinder, we obtain a rectangle with dimensions $2\pi k$ and $b - a$ (see Fig. 10.3.10), whose area is $2\pi k(b - a)$, so we can say that the area of the cylinder is $2\pi k(b - a)$.

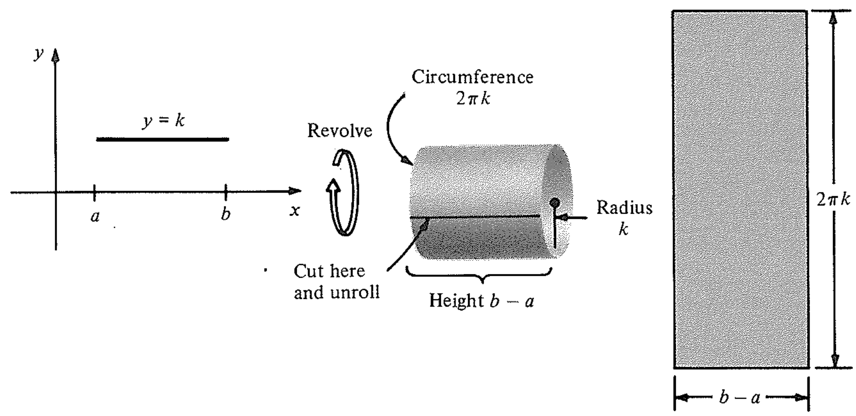


Figure 10.3.10. The area of the shaded cylinder is $2\pi k(b - a)$.

Next we look at the case where $f(x) = mx + q$, a linear function. The surface of revolution, as shown in Fig. 10.3.11, is a frustum of a cone—that is, the surface obtained from a right circular cone by cutting it with two planes perpendicular to the axis. To find the area of this surface, we may slit the frustum along a line and unroll it into the plane, as in Fig. 10.3.12, obtaining a circular sector of radius r and angle θ with a concentric sector of radius $r - s$ removed. By the definition of radian measure, we have $\theta r = 2\pi r_2$ and $\theta(r - s) = 2\pi r_1$, so $\theta s = 2\pi(r_2 - r_1)$, or $\theta = 2\pi[(r_2 - r_1)/s]$; from this we find that $r = r_2 s/(r_2 - r_1)$. The area of the figure is

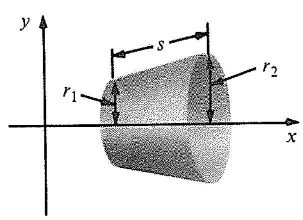


Figure 10.3.11. A frustum of a cone.

$$\begin{aligned}\frac{\theta}{2\pi}\left[\pi r^2 - \pi(r - s)^2\right] &= \frac{\theta}{2}\left[r^2 - (r^2 - 2rs + s^2)\right] = \frac{\theta}{2}(2rs - s^2)\\ &= \theta s\left(r - \frac{s}{2}\right) = 2\pi(r_2 - r_1)\left(\frac{r_2 s}{r_2 - r_1} - \frac{s}{2}\right)\\ &= 2\pi s\left(r_2 - \frac{r_2 - r_1}{2}\right) = \pi s(r_1 + r_2).\end{aligned}$$

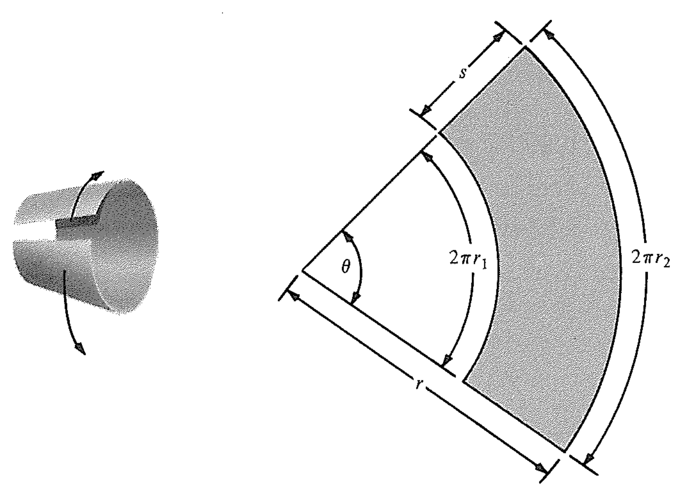


Figure 10.3.12. The area of the frustum, found by cutting and unrolling it, is $\pi s(r_1 + r_2)$.

(Notice that the proof breaks down in the case $r_1 = r_2 = k$, a cylinder, since r is then "infinite." Nevertheless, the resulting formula $2\pi ks$ for the area is still correct!)

We now wish to express $\pi s(r_1 + r_2)$ as an integral involving the function $f(x) = mx + q$. We have $r_1 = ma + q$, $r_2 = mb + q$, and $s = \sqrt{1+m^2}\,(b-a)$ (see Fig. 10.3.3), so the area is

$$\begin{aligned}
&\pi\sqrt{1+m^2}\,(b-a)\left[m(b+a)+2q\right]\\
&\quad= \pi\sqrt{1+m^2}\left[m(b^2-a^2)+2q(b-a)\right]\\
&\quad= 2\pi\sqrt{1+m^2}\left[m\frac{b^2-a^2}{2}+q(b-a)\right]\\
&\quad= 2\pi\sqrt{1+m^2}\left(m\frac{x^2}{2}+qx\right)\Bigg|_a^b .
\end{aligned}$$

Since $m(x^2/2) + qx$ is the antiderivative of $mx + q$, we have

$$\begin{aligned}
&2\pi\sqrt{1+m^2}\left(m\frac{x^2}{2}+qx\right)\Bigg|_a^b\\
&\quad= 2\pi\int_a^b \sqrt{1+m^2}\,(mx+q)\,dx\\
&\quad= 2\pi\int_a^b \left[\sqrt{1+f'(x)^2}\,\right] f(x)\,dx,
\end{aligned}$$

so we have succeeded in expressing the surface area as an integral.

Now we are ready to work with general surfaces. If $f(x)$ is *piecewise* linear on $[a, b]$, the surface obtained is a "conoid," produced by pasting together a finite sequence of frustums of cones, as in Fig. 10.3.13.

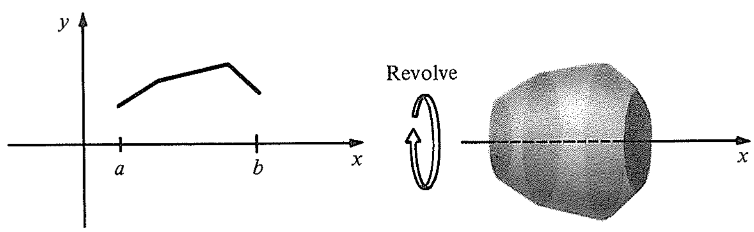


Figure 10.3.13. The surface obtained by revolving the graph of a piecewise linear function is a "conoid" consisting of several frustums pasted together.

The area of the conoid is the sum of the areas of the component frustums. Since the area of each frustum is given by the integral of the function $2\pi\sqrt{1+[f'(x)]^2}\,f(x)$ over the appropriate interval, the additivity of the integral implies that the area of the conoid is given by the same formula:

$$A = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx.$$

We now assert, as we did for arc length, that this formula is true for general functions f. [To do this rigorously, we would need a precise definition of surface area, which is rather complicated to give (much more complicated, even, than for arc length).]

Example 9 The polygonal line joining the points $(2,0)$, $(4,4)$, $(7,5)$, and $(8,3)$ is revolved about the x axis. Find the area of the resulting surface of revolution.

Solution The function f whose graph is the given polygon is

$$f(x)=\begin{cases}2(x-2), & 2\leqslant x\leqslant 4,\\ \frac{1}{3}(x-4)+4, & 4\leqslant x\leqslant 7,\\ -2(x-7)+5, & 7\leqslant x\leqslant 8.\end{cases}$$

Then we have

$$f'(x)=\begin{cases}2, & 2<x<4,\\ \frac{1}{3}, & 4<x<7,\\ -2, & 7<x<8.\end{cases}$$

Thus, $A=2\pi\int_2^8 f(x)\sqrt{1+f'(x)^2}\,dx$

$$=2\pi\left[\int_2^4 f(x)\sqrt{1+f'(x)^2}\,dx+\int_4^7 f(x)\sqrt{1+f'(x)^2}\,dx+\int_7^8 f(x)\sqrt{1+f'(x)^2}\,dx\right]$$

$$=2\pi\left\{\int_2^4[2(x-2)]\sqrt{1+4}\,dx+\int_4^7\left[\frac{1}{3}(x-4)+4\right]\sqrt{1+\frac{1}{9}}\,dx+\int_7^8[-2(x-7)+5]\sqrt{1+4}\,dx\right\}.$$

Using $\int(x-a)\,dx=\frac{1}{2}(x-a)^2+C$, we find

$$A=2\pi\left\{\sqrt{5}\left[(x-2)^2\right]\Big|_2^4+\frac{\sqrt{10}}{3}\left[\frac{1}{6}(x-4)^2+4x\right]\Big|_4^7+\sqrt{5}\left[-(x-7)^2+5x\right]\Big|_7^8\right\}$$

$$=2\pi\left(4\sqrt{5}+\frac{9}{2}\sqrt{10}+4\sqrt{5}\right)\approx 201.8.\ \blacktriangle$$

Exercises for Section 10.3

1. Find the length of the graph of the function $f(x)=x^4/8+1/4x^2$ on $[1,3]$.
2. Find the length of the graph of the function $f(x)=(x^4-12x+3)/6x$ on $[2,4]$.
3. Find the length of the graph of the function $y=[x^3+(3/x)]/6$ on $1\leqslant x\leqslant 3$.
4. Find the length of the graph of the function $y=\sqrt{x}(4x-3)/6$ on $1\leqslant x\leqslant 9$.
5. Express the length of the graph of x^n on $[a,b]$ as an integral. (Do not evaluate.)
6. Express the length of the graph of $f(x)=\sin x$ on $[0,2\pi]$ as an integral. (Do not evaluate.)
7. Express the length of the graph of $f(x)=x\cos x$ on $[0,1]$ as an integral. (Do not evaluate.)
8. Express the length of the graph $y=e^{-x}$ on $[-1,1]$ as an integral. (Do not evaluate.)

In Exercises 9–12, let the graph of f consist of straight line segments joining the given points. Verify that the length of the graph as computed directly is equal to that given by the arc length formula.

9. $(0,0)$ to $(1,2)$ to $(2,1)$ to $(5,0)$.
10. $(1,1)$ to $(2,2)$ to $(3,0)$.
11. $(-1,-1)$ to $(0,1)$ to $(1,2)$ to $(2,-2)$.
12. $(-2,2)$ to $(-1,-3)$ to $(3,1)$.

Find the area of the surfaces obtained by revolving the curves in Exercises 13–20.

13. The graph of $\sqrt{x+1}$ on $[0,2]$ about the x axis.
14. The graph of $y=[x^3+(3/x)]/6$, $1\leqslant x\leqslant 3$ about the x axis.
15. The graph of $y=\sqrt{x}(4x-3)/6$, $1\leqslant x\leqslant 9$ about the x axis.
16. The graph of e^x on $[0,1]$ about the x axis.
17. The graph of $y=\cos x$ on $[-\pi/2,\pi/2]$ about the x axis.
18. The parabola $y=\sqrt{x}$ on $[4,5]$ about the x axis.
19. The graph of $x^{1/3}$ on $[1,3]$ about the y axis.
20. The graph of $y=\ln x$ on $[2,3]$ about the y axis.

In Exercises 21–24, the polygon joining the given points is revolved about the x axis. Find the area of the resulting surface of revolution.

21. $(0,0)$ to $(1,1)$ to $(2,0)$.
22. $(1,0)$ to $(3,2)$ to $(4,0)$.
23. $(2,1)$ to $(3,2)$ to $(4,1)$ to $(5,3)$.
24. $(4,0)$ to $(5,2)$ to $(6,1)$ to $(8,0)$.

25. Find the length of the graph of $a(x+b)^{3/2}+c$ on $[0,1]$, where a, b, and c are constants. What is the effect of changing the value of c?
26. Find the length of the graph of $y=x^2$ on $[0,b]$.
27. Express the length of the graph of $f(x)=2x^3$ on $[-1,2]$ as an integral. Evaluate numerically to within 1.0 by finding upper and lower sums. Compare your results with a string-and-ruler measurement.
28. Find the length, accurate to within 1 centimeter, of the curve in Fig. 10.3.14.

Figure 10.3.14. Find the length of this curve.

For each of the functions and intervals in Exercises 29–32, express as an integral: (a) the length of the curve; (b) the area of the surface obtained by revolving the curve about the x axis. (Do not evaluate the integrals.)

29. $\tan x + 2x$ on $[0, \pi/2]$
30. x^3+2x-1 on $[1,3]$
31. $1/x + x$ on $[1,2]$
32. $e^x + x^3$ on $[0,1]$

33. Find the area, accurate to within 5 square centimeters, of the surface obtained by revolving the curve in Fig. 10.3.15 around the x axis.

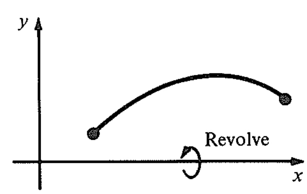


Figure 10.3.15. Find the area of the surface obtained by revolving this curve.

34. Use upper and lower sums to find the area, accurate to within 1 unit, of the surface obtained by revolving the graph of x^4 on $[0,1]$ about the x axis.
35. Prove that the length of the graph of $f(x)=\cos(\sqrt{3}\,x)$ on $[0,2\pi]$ is less than or equal to 4π.
36. Suppose that $f(x) \geqslant g(x)$ for all x in $[a,b]$. Does this imply that the length of the graph of f on $[a,b]$ is greater than or equal to that for g? Justify your answer by a proof or an example.

★37. Show that the length of the graph of $\sin x$ on $[0.1, 1]$ is less than the length of the graph of $1+x^4$ on $[0.1, 1]$.

★38. Suppose that the function f on $[a,b]$ has an inverse function g defined on $[\alpha,\beta]$. Assume that $0<a<b$ and $0<\alpha<\beta$.
   (a) Find a formula, in terms of f, for the area of the surface obtained by revolving the graph of g on $[\alpha,\beta]$ about the x axis.
   (b) Show that this formula is consistent with the one in formula (3) for the area of the surface obtained by revolving the graph of f on $[a,b]$ about the y axis.

★39. Write an integral representing the area of the surface obtained by revolving the graph of $1/(1+x^2)$ about the x axis. Do not evaluate the integral, but show that it is less than $2\sqrt{5}\,\pi^2$ no matter how long an interval is taken.

★40. Craftsman Cabinet Company was preparing a bid on a job that required epoxy coating of several tank interiors. The tanks were constructed from steel cylinders C feet in circumference and height H feet, with spherical steel caps welded to each end (see Fig. 10.3.16). Specifications required a $\frac{1}{8}$-inch coating. The 20-year-old estimator quickly figured the cylindrical part as HC square feet. For the spherical cap he stretched a tape measure over the cap to obtain S ft.

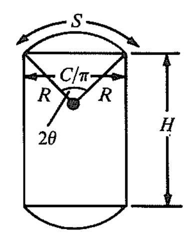


Figure 10.3.16. A cross section of a tank requiring an epoxy coating on its interior.

   (a) Write equations which can be used to find the surface area of the steel cap in terms of S and C. [*Hint*: Revolve $y=\sqrt{R^2-x^2}$ about the y axis, $0 \leqslant x \leqslant C/2\pi$.]
   (b) Write equations for finding the surface area of the tank in terms of S, C, and H.
   (c) Determine the cost for six tanks with $H=16$ feet, $C=37.7$ feet, $S=13.2$ feet, given that the coating costs \$2.10 per square foot.

★41. (a) Calculate the area of the frustum shown in Fig. 10.3.17 using geometry alone. (b) Derive formula (3) using step functions.

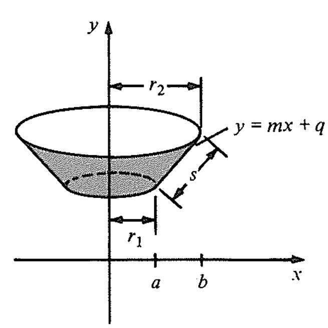


Figure 10.3.17. A line segment revolved around the y axis becomes a frustum of a cone.

10.4 Parametric Curves

Arc lengths may be found by integral calculus for curves which are not graphs of functions.

We begin this section with a study of the differential calculus of parametric curves, a topic which was introduced in Section 2.4. The arc length of a parametric curve is then expressed as an integral.

Recall from Section 2.4 that a *parametric curve* in the xy plane is specified by a pair of functions: $x = f(t)$, $y = g(t)$. The variable t, called the *parameter* of the curve, may be thought of as time; the pair $(f(t), g(t))$ then describes the path in the plane of a moving point. Many physical situations, such as the motion of the Earth about the sun and a car moving on a twisting highway, can be conveniently idealized as parametric curves.

Example 1 (a) Describe the motion of the point (x, y) if $x = \cos t$ and $y = \sin t$, for t in $[0, 2\pi]$. (b) Describe the motion of the point (t, t^3) for t in $(-\infty, \infty)$.

Solution (a) At $t = 0$, the point is at $(1, 0)$. Since $\cos^2 t + \sin^2 t = 1$, the point (x, y) satisfies $x^2 + y^2 = 1$, so it moves on the unit circle. As t increases from zero, $x = \cos t$ decreases and $y = \sin t$ increases, so the point moves in a counterclockwise direction. Finally, since $(\cos(2\pi), \sin(2\pi)) = (1, 0)$, the point makes a full rotation after 2π units of time (see Fig. 10.4.1).
(b) We have $x^3 = t^3 = y$, so the point is on the curve $y = x^3$. As t increases so does x, and the point moves from left to right (see Fig. 10.4.2). ▲

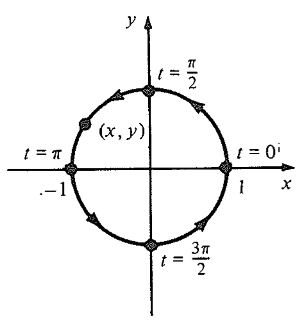


Figure 10.4.1. The point $(\cos t, \sin t)$ moves in a circle.

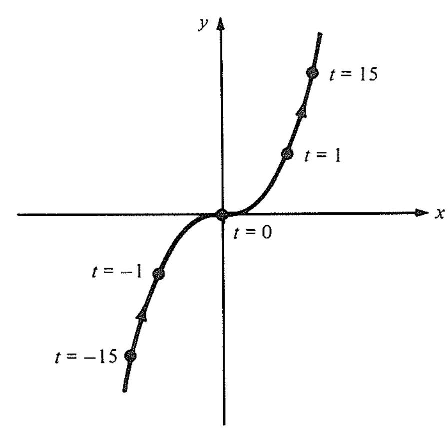


Figure 10.4.2. The motion of the point (t, t^3).

Example 1(b) illustrates a general fact: Any curve $y = f(x)$ which is the graph of a function can be described parametrically: we set $x = t$ and $y = f(t)$. However, parametric equations can describe curves which are not the graphs of functions, like the circle in Example 1(a).

The equations

$$x = at + b, \qquad y = ct + d$$

describe a straight line. To show this, we *eliminate the parameter t* in the following way. If $a \neq 0$, solve the first equation for t, getting $t = (x - b)/a$. Substituting this into the second equation gives $y = c[(x - b)/a] + d$; that is, $y = (c/a)x + (ad - bc)/a$, which is a straight line with slope c/a. If $a = 0$, we have $x = b$ and $y = ct + d$. If $c \neq 0$, then y takes all values as t varies and b is fixed, so we have the vertical line $x = b$ (which is not the graph of a function). If $c = 0$ as well as $a = 0$, then $x = b$ and $y = d$, so the graph is a "stationary" point (b, d).

Similarly, we can see that

$$x = r\cos t + x_0, \qquad y = r\sin t + y_0$$

describes a circle by writing

$$\frac{x - x_0}{r} = \cos t, \qquad \frac{y - y_0}{r} = \sin t.$$

Therefore,

$$\left(\frac{x - x_0}{r}\right)^2 + \left(\frac{y - y_0}{r}\right)^2 = \cos^2 t + \sin^2 t = 1$$

or $(x - x_0)^2 + (y - y_0)^2 = r^2$, which is the equation of a circle with radius r and center (x_0, y_0). As t varies from 0 to 2π, the point (x, y) moves once around the circle.

Parametric Equations of Lines and Circles

Straight line

$x = at + b,$	$-\infty < t < \infty;$
$y = ct + d$	a and c not both zero; the line passes through (b, d) with slope c/a.

Circle

$x = r\cos t + x_0,$	$0 \leqslant t \leqslant 2\pi;$
$y = r\sin t + y_0,$	$r > 0$, $r =$ radius, $(x_0, y_0) =$ center.

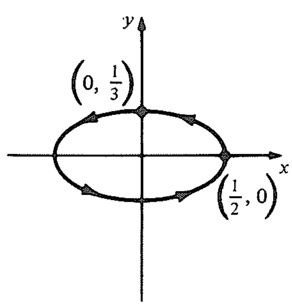


Figure 10.4.3. The parametric curve $x = \frac{1}{2}\cos t, y = \frac{1}{3}\sin t$ is an ellipse.

Other curves can be written conveniently in parametric form as well. For example, $4x^2 + 9y^2 = 1$ (an ellipse) can be written as $x = \frac{1}{2}\cos t$, $y = \frac{1}{3}\sin t$. As t goes from 0 to 2π, the point moves once around the ellipse (see Fig. 10.4.3). General properties of ellipses are studied in Section 14.1.

The same geometric curve can often be represented parametrically in more than one way. For example, the line $x = at + b$, $y = ct + d$ can also be represented by

$$x = t, \qquad y = \frac{ct}{a} + \frac{ad - bc}{a}$$

or by

$$x = t^3, \qquad y = \frac{ct^3}{a} + \frac{ad - bc}{a}.$$

(If we used t^2, we would get only half of the line since $t^2 \geqslant 0$ for all t.)

In Section 2.4 we saw that the tangent line to a parametric curve $(x, y) = (f(t), g(t))$ at the point $(f(t_0), g(t_0))$ has slope

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t_0)}{f'(t_0)}$$

If $f'(t_0) = 0$ and $g'(t_0) \neq 0$, the tangent line is vertical; if $f'(t_0)$ and $g'(t_0)$ are both zero, the tangent line is not defined. Since the tangent line passes through $(f(t_0), g(t_0))$, we may write its equation in point-slope form:

$$y = \frac{g'(t_0)}{f'(t_0)}[x - f(t_0)] + g(t_0). \tag{1}$$

Example 2 Find the equation of the tangent line when $t = 1$ for the curve $x = t^4 + 2\sqrt{t}$, $y = \sin(t\pi)$.

Solution When $t = 1$, $x = 3$ and $y = \sin\pi = 0$. Furthermore, $dx/dt = 4t^3 + 1/\sqrt{t}$, which equals 5 when $t = 1$; $dy/dt = \pi\cos(t\pi)$, which equals $-\pi$ when $t = 1$. Thus the equation of the tangent line is, by formula (1),

$$y = -\frac{\pi}{5}(x - 3) + 0 \quad \text{or} \quad y = -\frac{\pi}{5}x + \frac{3\pi}{5}. \blacktriangle$$

If a curve is given parametrically, it is natural to express its tangent line parametrically as well. To do this, we transform equation (1) to the form

$$\frac{y - g(t_0)}{g'(t_0)} = \frac{x - f(t_0)}{f'(t_0)}.$$

We can set both sides of this equation equal to t, obtaining

$$x = tf'(t_0) + f(t_0), \qquad y = tg'(t_0) + g(t_0). \tag{2}$$

Equation (2) is the parametric equation for a line with slope $g'(t_0)/f'(t_0)$ if $f'(t_0) \neq 0$. If $f'(t_0) = 0$ but $g'(t_0) \neq 0$, equations (2) describe a vertical line. If $f'(t_0)$ and $g'(t_0)$ are both zero, equations (2) describe a stationary point.

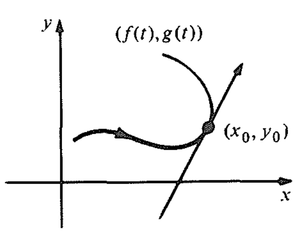


Figure 10.4.4. If the forces constraining a particle to the curve $(f(t), g(t))$ are removed at t_0, then the particle will follow the tangent line at t_0.

It is convenient to make one more transformation of equations (2), so that the tangent line passes through (x_0, y_0) at the same time t_0 as the curve, rather than at $t = 0$. Substituting $t - t_0$ for t, we obtain the formulas

$$x = f'(t_0)(t - t_0) + f(t_0), \qquad y = g'(t_0)(t - t_0) + g(t_0). \tag{3}$$

Notice that the functions in formulas (3) which define the tangent line to a curve are exactly the *linear approximations* to the functions defining the curve itself. If we think of $(x, y) = (f(t), g(t))$ as the position of a moving particle, then the tangent line at t_0 is the path which the particle would follow if, at time t_0, all constraining forces were suddenly removed and the particle were allowed to move freely in a straight line. (See Fig. 10.4.4.)

Example 3 A child is whirling an object on a string, letting out string at a constant rate, so that the object follows the path $x = (1 + t)\cos t$, $y = (1 + t)\sin t$.

(a) Sketch the path for $0 \leqslant t \leqslant 4\pi$.
(b) At $t = 4\pi$ the string breaks, so that the object follows its tangent line. Where is the object at $t = 5\pi$?

Solution (a) By plotting some points and thinking of (x, y) as moving in an ever enlarging circle, we obtain the sketch in Fig. 10.4.5.

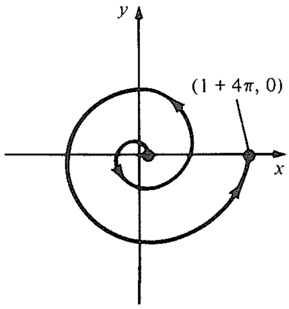


Figure 10.4.5. The curve $((1+t)\cos t, (1+t)\sin t)$ for t in $[0, 4\pi]$.

(b) We differentiate:

$$f'(t) = \frac{dx}{dt} = (1+t)(-\sin t) + \cos t, \quad \text{and}$$

$$g'(t) = \frac{dy}{dt} = (1+t)\cos t + \sin t.$$

When $t_0 = 4\pi$, we have

$$f(t_0) = (1+4\pi)\cos 4\pi = 1 + 4\pi, \quad \text{and} \quad g(t_0) = (1+4\pi)\sin 4\pi = 0,$$

$$f'(t_0) = (1+4\pi)\cdot 0 + 1 = 1, \quad \text{and} \quad g'(t_0) = (1+4\pi)\cdot 1 + 0 = 1 + 4\pi.$$

By formulas (3), the equations of the tangent line are

$$x = t - 4\pi + (1+4\pi), \qquad y = (1+4\pi)(t - 4\pi) + 0.$$

When $t = 5\pi$, the object, which is now following the tangent line, is at $x = 1 + 5\pi \approx 16.71$, $y = (1+4\pi)\pi \approx 42.62$. ▲

Tangents to Parametric Curves

Let $x = f(t)$ and $y = g(t)$ be the parametric equations of a curve C. If f and g are differentiable at t_0, and $f'(t_0)$ and $g'(t_0)$ are not both zero, then the tangent line to C at t_0 is defined by the parametric equations:

$$x = f'(t_0)(t - t_0) + f(t_0), \qquad y = g'(t_0)(t - t_0) + g(t_0).$$

If $f'(t_0) \neq 0$, this line has slope $g'(t_0)/f'(t_0)$, and its equation can be written as

$$y = \frac{g'(t_0)}{f'(t_0)}\left[x - f(t_0)\right] + g(t_0).$$

If $f'(t_0) = 0$ and $g'(t_0) \neq 0$, the line is vertical; its equation is

$$x = f(t_0).$$

Example 4 Consider the curve $x = t^3 - t$, $y = t^2$.

(a) Plot the points corresponding to $t = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2$.
(b) Using these points, together with the behavior of the functions t^2 and $t^3 - t$, sketch the entire curve.
(c) Find the slope of the tangent line at the points corresponding to $t = 1$ and $t = -1$.
(d) Eliminate the parameter t to obtain an equation in x and y for the curve.

Solution (a) We begin by making a table:

t	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
$x = t^3 - t$	-6	0	$\frac{3}{8}$	0	$-\frac{3}{8}$	0	6
$y = t^2$	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4

These points are plotted in Fig. 10.4.6. The number next to each point is the corresponding value of t. Notice that the point $(0, 1)$ occurs for $t = -1$ and $t = 1$.

(b) We plot x and y against t in Fig. 10.4.7. From the graph of x against t, we conclude that as t goes from $-\infty$ to ∞, the point comes in from the left, reverses direction for a while, and then goes out to the right. From the graph

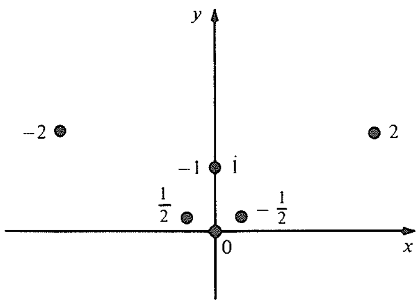


Figure 10.4.6. Some points on the curve $(t^3 - t, t^2)$.

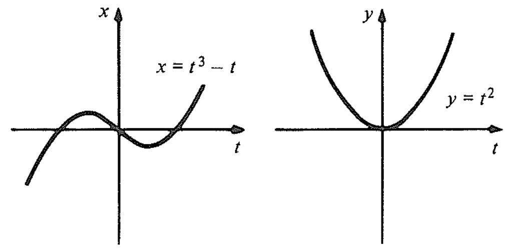


Figure 10.4.7. The graphs of x and y plotted separately against t.

of y against t, we see that the point descends for $t < 0$, reaches the bottom at $y = 0$ when $t = 0$, and then ascends for $t > 0$. Putting this information together with the points we have plotted, we sketch the curve in Fig. 10.4.8.

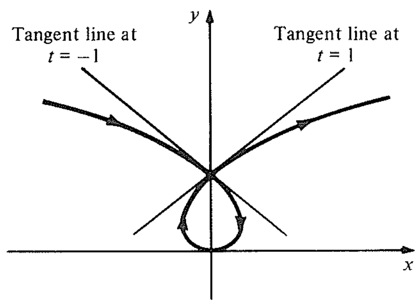


Figure 10.4.8. The parametric curve $(t^3 - t, t^2)$.

(c) The slope of the tangent line at time t is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2 - 1}.$$

When $t = -1$, the slope is -1; when $t = 1$, the slope is 1. (See Fig. 10.4.8.)

(d) We can eliminate t by solving the second equation for t to get $t = \pm\sqrt{y}$ and substituting in the first to get $x = \pm(y^{3/2} - y^{1/2})$. To obtain an equation without fractional powers, we square both sides. The result is $x^2 = y(y-1)^2$, or $x^2 = y^3 - 2y^2 + y$. In this form, it is not so easy to predict the behavior of the curve, particularly at the "double point" $(0, 1)$. ▲

Example 5 (a) Sketch the curve $x = t^3, y = t^2$. (b) Find the equation of the tangent line at $t = 1$. (c) What happens at $t = 0$?

Solution (a) Eliminating the parameter t, we have $y = x^{2/3}$. The graph has a cusp at the origin, as in Fig. 10.4.9. (Cusps were discussed in Section 3.4.)

(b) When $t = 1$, we have $x = t^3 = 1$, $y = t^2 = 1$, $dx/dt = 3t^2 = 3$, and $dy/dt = 2t = 2$, so the tangent line is given by

$$x = 3(t-1) + 1, \qquad y = 2(t-1) + 1.$$

It has slope $\frac{2}{3}$. (You can also see this by differentiating $y = x^{2/3}$ and setting $x = 1$.)

(c) When $t = 0$, we have $dx/dt = 0$ and $dy/dt = 0$, so the tangent line is not defined. ▲

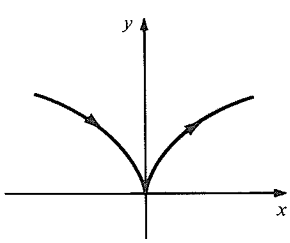


Figure 10.4.9. The curve (t^3, t^2) has a cusp at the origin.

Example 6 Consider the curve $x = \cos 3t$, $y = \sin t$. Find the points where the tangent is horizontal and those where it is vertical. Use this information to sketch the curve.

Solution The tangent line is vertical when $dx/dt = 0$ and horizontal when $dy/dt = 0$. (If both are zero, there is no tangent line.)

We have $dx/dt = -3\sin 3t$, which is zero when $t = 0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3$ (the curve repeats itself when t reaches 2π); $dy/dt = \cos t$, which is zero when $t = \pi/2$ or $3\pi/2$. We make a table:

t	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$
$x = \cos 3t$	1	-1	0	1	-1	1	0	-1
$y = \sin t$	0	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$
Tangent	vert	vert	hor	vert	vert	vert	hor	vert

Using the fact that $\sqrt{3}/2 \approx 0.866$, we sketch this information in Fig. 10.4.10. Connecting these points in the proper order with a smooth curve, we obtain Fig. 10.4.11. This curve is an example of a *Lissajous figure* (see Review Exercise 93 and 94 at the end of this chapter). ▲

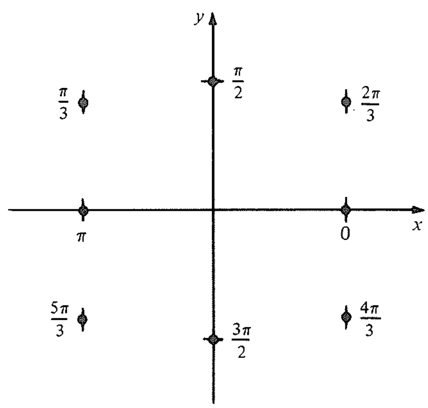


Figure 10.4.10. Points on the curve $(\cos 3t, \sin t)$ with horizontal and vertical tangent.

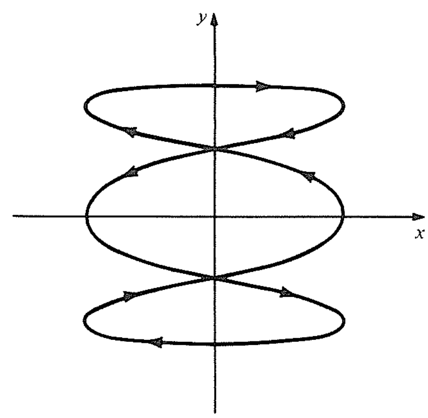


Figure 10.4.11. The curve $(\cos 3t, \sin t)$ is an example of a Lissajous figure.

What is the length of the curve given by $(x, y) = (f(t), g(t))$ for $a \leqslant t \leqslant b$? To get a formula in terms of f and g, we begin by considering the case in which the point $(f(t), g(t))$ moves along the graph of a function $y = h(x)$; that is, $g(t) = h(f(t))$.

If $f(a) = \alpha$ and $f(b) = \beta$, the length of the curve is $\int_\alpha^\beta \sqrt{1 + [h'(x)]^2}\, dx$ by formula (1) of Section 10.3. If we change variables from x to t in this integral, we have $dx = f'(t)\,dt$, so the length is

$$\int_a^b \sqrt{1 + [h'(f(t))]^2}\, f'(t)\, dt.$$

To eliminate the function h from this formula, we may apply the chain rule to

$g(t) = h(f(t))$, getting $g'(t) = h'(f(t)) \cdot f'(t)$. Solving for $h'(f(t))$ and substituting in the integral gives

$$\int_a^b \sqrt{1 + \left[\frac{g'(t)}{f'(t)}\right]^2}\, f'(t)\, dt$$

or

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{4}$$

Formula (4) involves only the information contained in the parametrization. Since we can break up any reasonably behaved parametric curve into segments, each of which is the graph of a function or a vertical line (for which we see that equation (4) gives the correct length, since $f'(t) \equiv 0$), we conclude that equation (4) ought to be valid for any parametric curve.

Equation (4) may be derived using infinitesimals in the following way. Refer to Fig. 10.4.12 and note that $ds^2 = dx^2 + dy^2$. Thus

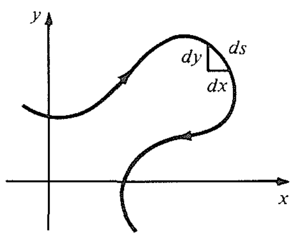


Figure 10.4.12. Finding the length of a parametric curve.

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Integrating from $t = a$ to $t = b$ reproduces formula (4).

Length of a Parametric Curve

Suppose that a parametric curve C is given by continuous functions $x = f(t)$, $y = g(t)$, for $a \leqslant t \leqslant b$, and that $f'(t)$ and $g'(t)$ exist and are continuous, except possibly for finitely many points. Then the length of C is given by

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example 7 Find the length of the circle of radius 2 which is given by the parametric equations $x = 2\cos t + 3$, $y = 2\sin t + 4$, $0 \leqslant t \leqslant 2\pi$.

Solution We find $f'(t) = dx/dt = -2\sin t$ and $g'(t) = dy/dt = 2\cos t$, so

$$L = \int_0^{2\pi} \sqrt{4\sin^2 t + 4\cos^2 t}\, dt$$
$$= \int_0^{2\pi} 2\sqrt{\sin^2 t + \cos^2 t}\, dt = \int_0^{2\pi} 2\, dt = 4\pi$$

(which equals 2π times the radius). ▲

Example 8 Find the length of (a) $x = t^8$, $y = t^4$ on $[1,3]$ and (b) $x = t\sin t$, $y = t\cos t$ on $[0, 4\pi]$.

Solution (a) We are given $x = f(t) = t^8$ and $y = g(t) = t^4$ on $[1,3]$. The length is

$$L = \int_1^3 \sqrt{(8t^7)^2 + (4t^3)^2}\, dt = \int_1^3 4t^3\sqrt{(2t^4)^2 + 1}\, dt.$$

Letting $u = 2t^4$, we have the length

$$L = \frac{1}{2}\int_2^{162}\sqrt{u^2+1}\,du.$$

Making the substitution $u = \tan\theta$, $du = \sec^2\theta\,d\theta$, we get

$$\int\sqrt{u^2+1}\,du = \int\sec^3\theta\,d\theta = I.$$

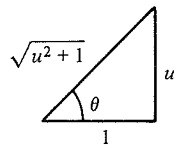


Figure 10.4.13. If $\tan\theta = u$, $\sqrt{u^2+1} = \sec\theta$.

(see Fig. 10.4.13). Integrating by parts,

$$I = \sec\theta\tan\theta - \int\sec\theta\tan^2\theta\,d\theta = \sec\theta\tan\theta - \int\sec\theta(\sec^2\theta - 1)\,d\theta.$$

Since

$$\int\sec\theta\,d\theta = \int\sec\theta\cdot\frac{\tan\theta+\sec\theta}{\tan\theta+\sec\theta}\,d\theta = \ln|\tan\theta+\sec\theta|,$$

we get $I = \sec\theta\tan\theta - I + \ln|\tan\theta+\sec\theta| + C$. Thus

$$I = \int\sec^3\theta\,d\theta = \frac{\sec\theta\tan\theta + \ln|\tan\theta+\sec\theta|}{2} + C.$$

(Compare Example 3, Section 10.3.) Putting everything together in terms of u,

$$L = \frac{1}{4}\left[\sqrt{u^2+1}\cdot u + \ln\left|u+\sqrt{u^2+1}\right|\right]\Bigg|_2^{162} \approx 6561.1.$$

(b) If $x = t\sin t$ and $y = t\cos t$, $dx/dt = \sin t + t\cos t$ and $dy/dt = \cos t - t\sin t$. Therefore,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t$$
$$= 1 + t^2.$$

Thus, using (a), the length is

$$\int_0^{4\pi}\sqrt{1+t^2}\,dt = \frac{1}{2}\left[t\sqrt{1+t^2} + \ln\left|t+\sqrt{1+t^2}\right|\right]\Bigg|_0^{4\pi}$$
$$= \frac{1}{2}\left[4\pi\sqrt{1+16\pi^2} + \ln\left(4\pi+\sqrt{1+16\pi^2}\right)\right] \approx 80.8 \blacktriangle$$

Example 9 Show that if $x = f(t)$ and $y = g(t)$ is any curve with $(f(0), g(0)) = (0,0)$ and $(f(1), g(1)) = (0,a)$, then the length of the curve for $0 \leqslant t \leqslant 1$ is at least equal to a. What can you say if the length is exactly equal to a?

Solution It is evident that $[g'(t)]^2 \leqslant [f'(t)]^2 + [g'(t)]^2$, so

$$g'(t) \leqslant \sqrt{[f'(t)]^2 + [g'(t)]^2}\,.$$

Integrating from 0 to 1, we have

$$\int_0^1 g'(t)\,dt \leqslant \int_0^1\sqrt{[f'(t)]^2+[g'(t)]^2}\,dt.$$

By the fundamental theorem of calculus, the left-hand side is equal to $g(1) - g(0) = a - 0 = a$; the right-hand side is the length L of the curve, so we have $a \leqslant L$. If $a = L$, the integrands must be equal; that is, $g'(t) = \sqrt{f'(t)^2 + g'(t)^2}$, which is possible only if $f'(t)$ is identically zero; that is, $f(t)$ is constant. Since $f(0) = f(1) = 0$, we must have $f(t)$ identically zero; that is, the point (x, y) stays on the y axis.

We have shown that the shortest curve between the points $(0,0)$ and $(0,a)$ is the straight line segment which joins them. $\blacktriangle$

Given a point moving according to $x = f(t)$, $y = g(t)$, the integral

$$D(t) = \int_a^t \sqrt{[f'(s)]^2 + [g'(s)]^2}\, ds$$

is the distance (along the curve) travelled by the point between time a and time t. The derivative $D'(t)$ should then represent the speed of the point along the curve. By the fundamental theorem of calculus (alternative version), we have

$$D'(t) = \sqrt{[f'(t)]^2 + [g'(t)]^2}\,.$$

Speed

Let a point move according to the equations $x = f(t)$, $y = g(t)$. Then the speed of the point at time t is

$$\sqrt{[f'(t)]^2 + [g'(t)]^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,.$$

Suppose that an object is constrained to move along the curve $x = f(t)$, $y = g(t)$ and that at time t_0 the constraining forces are removed, so the particle continues along the tangent line

$$x = f'(t_0)(t - t_0) + f(t_0), \qquad y = g'(t_0)(t - t_0) + g(t_0).$$

At time $t_0 + \Delta t$, the particle is at $(f'(t_0)\Delta t + f(t_0),\ g'(t_0)\Delta t + g(t_0))$, which is at distance $\sqrt{f'(t_0)^2 + g'(t_0)^2}\,\Delta t$ from $(f(t_0), g(t_0))$. Thus the distance travelled in time Δt after the force is removed is equal to Δt times the speed at t_0, so we have another justification of our formula for the speed.

Example 10 A particle moves around the elliptical track $4x^2 + y^2 = 4$ according to the equations $x = \cos t$, $y = 2\sin t$. When is the speed greatest? Where is it least?

Solution The speed is

$$\sqrt{\left[\frac{d(\cos t)}{dt}\right]^2 + \left[\frac{d(2\sin t)}{dt}\right]^2} = \sqrt{\sin^2 t + 4\cos^2 t} = \sqrt{1 + 3\cos^2 t}\,.$$

Without any further calculus, we observe that the speed is greatest when $\cos t = \pm 1$; that is, $t = 0$, π, 2π, and so forth. The speed is least when $\cos t = 0$; that is, $t = \pi/2$, $3\pi/2$, $5\pi/2$, and so on. ▲

Example 11 The position (x, y) of a bulge in a bicycle tire as it rolls down the street can be parametrized by the angle θ shown in Fig. 10.4.14. Let the radius of the tire be a. It can be verified by methods of plane trigonometry that $x = a\theta - a\sin\theta$, $y = a - a\cos\theta$. (This curve is called a *cycloid*.)

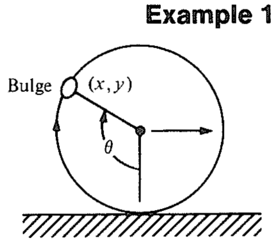


Figure 10.4.14. Investigate how a bulge on a tire moves.

(a) Find the distance travelled by the bulge for $0 \leqslant \theta \leqslant 2\pi$, using the identity $1 - \cos\theta = 2\sin^2(\theta/2)$. This distance is greater than $2\pi a$ (distance the tire rolls).

(b) Draw a figure for one arch of the cycloid, and superimpose the circle of radius a with center at $(\pi a, a)$, together with the line segment $0 \leqslant x \leqslant 2\pi a$ on the x axis. Show that the three enclosed areas are each πa^2.

Solution (a) The distance d is the arc length of the cycloid for $0 \leqslant \theta \leqslant 2\pi$. Thus,

$$d = \int_0^{2\pi} \sqrt{(a - a\cos\theta)^2 + (a\sin\theta)^2}\, d\theta$$
$$= a\int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta = a\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos\theta}\, d\theta$$
$$= a\sqrt{2}\int_0^{2\pi}\sqrt{2}\sin\left(\frac{\theta}{2}\right) d\theta = -4a\cos(\theta/2)\Big|_0^{2\pi} = -4a(-1-1) = 8a.$$

(b) Refer to Fig. 10.4.15. The total area beneath the arch is

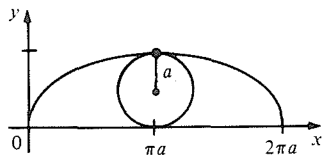


Figure 10.4.15. One arch of the cycloid.

$$A = \int_0^{2\pi a} y\, dx = \int_0^{2\pi} y\frac{dx}{d\theta}\, d\theta$$
$$= \int_0^{2\pi} a^2(1-\cos\theta)(1-\cos\theta)\, d\theta = a^2\int_0^{2\pi}(1 - 2\cos\theta + \cos^2\theta)\, d\theta$$
$$= a^2\left[(\theta - 2\sin\theta)\Big|_0^{2\pi} + \frac{1}{2}\int_0^{2\pi}(1+\cos 2\theta)\, d\theta\right]$$
$$= a^2\left[2\pi + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}\right] = 3\pi a^2.$$

The area of the circle is πa^2, so by symmetry each of the other two congruent regions also has area πa^2. ▲

Exercises for Section 10.4

For the parametric curves in Exercises 1–4, sketch the curve and find an equation in x and y by eliminating the parameter.

1. $x = 4t - 1, y = t + 2$.
2. $x = 2t + 1, y = t^2$.
3. $x = \cos\theta + 1, y = \sin\theta$.
4. $x = \sin\theta, y = \cos\theta - 3$.

Find a parametric representation for each of the curves in Exercises 5–12.

5. $2x^2 + y^2 = 1$.
6. $16x^2 + 9y^2 = 1$.
7. $4xy = 1$.
8. $y = 3x - 2$.
9. $y = x^3 + 1$.
10. $3x^2 - y^2 = 1$.
11. $y = \cos(2x)$.
12. $y^2 = x + x^2$.

Find the equation of the tangent line to each of the curves in Exercises 13–16 at the given point.

13. $x = \frac{1}{2}t^2 + t, y = t^{2/3}$; $t_0 = 1$.
14. $x = 1/t, y = \sqrt{t+1}$; $t_0 = 2$.
15. $x = \cos^2(t/2), y = \frac{1}{2}\sin t$; $t_0 = \pi/2$.
16. $x = \theta - \sin\theta, y = 1 - \cos\theta$; $\theta_0 = \pi/4$.

17. A bead is sliding on a wire, having position $x = (2-3t)^2$, $y = 2 - 3t$ at time t. If the bead flies off the wire at time $t = 1$, where is it when $t = 3$?
18. A piece of mud on a bicycle tire is following the cycloid $x = 6t - 3\sin 2t$, $y = 3 - 3\cos 2t$. At time $t = \pi/2$, the mud becomes detached from the tire. Along what line is it moving? (Ignore gravity.)

Sketch each of the parametric curves in Exercises 19–22, find an equation in x and y by eliminating the parameter, and find the points where the tangent line is horizontal or vertical.

19. $x = t^2$; $y = \cos t$. (What happens at $t = 0$?)
20. $x = \pi/2 - s$; $y = 2\sin 2s$.
21. $x = \cos 2t$; $y = \sin t$.
22. $(x, y) = (\cos t, \sin 2t)$.

23. Find the length of $x = t^2, y = t^3$ on $[0, 1]$.
24. Find the length of the curve given by $x = \frac{1}{2}\sin 2t$, $y = 3 + \cos^2 t$ on $[0, \pi]$.
25. Find the length of the curve (t^2, t^4) on $0 \leqslant t \leqslant 1$.
26. Find the length of the parametric curve $(e^t(\cos t)\sqrt{t^3+1}, 2e^t(\cos t)\sqrt{t^3+1})$ on $[0, 1]$.
27. Show that if $x = a\cos t + b$ and $y = a\sin t + d$: (a) the speed is constant; (b) the length of the curve on $[t_0, t_1]$ is equal to the speed times the elapsed time $(t_1 - t_0)$.
28. An object moves from left to right along the curve $y = x^{3/2}$ at constant speed. If the point is at $(0, 0)$ at noon and at $(1, 1)$ at 1:00 P.M., where is it at 1:30 P.M.?
29. Consider the parametrized curve $x = 2\cos\theta$, $y = \theta - \sin\theta$.
    (a) Find the equation of the tangent line at $\theta = \pi/2$.
    (b) Sketch the curve.
    (c) Express the length of the curve on $[0, \pi]$ as an integral.

30. Show that if
$$\frac{dx}{dt}\frac{d^2x}{dt^2} = -\frac{dy}{dt}\frac{d^2y}{dt^2},$$
then the speed of the curve $x = f(t)$, $y = g(t)$ is constant.

31. A particle travels a path in space with speed $s(t) = \sin^2(\pi t) + \tan^4(\pi t)\sec^2(\pi t)$. Find the distance $\int_0^{10} s(t)\,dt$ travelled in the first ten seconds.

32. A car loaded with skiers climbs a hill to a ski resort, constantly changing gears due to variations in the incline. Assume, for simplicity, that the motion of the auto is planar: $x = x(t)$, and $y = y(t)$, $0 \leqslant t \leqslant T$. Let $s(t)$ be the distance travelled along the road at time t (Fig. 10.4.16).

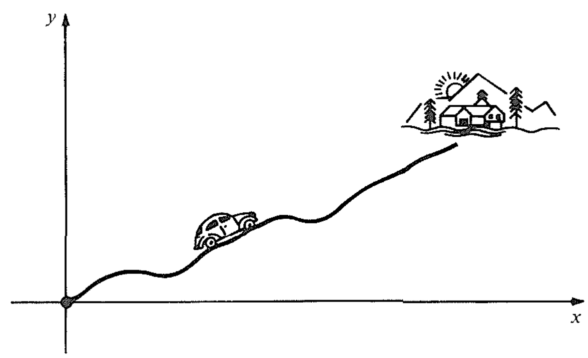


Figure 10.4.16. A car on its way to a ski cabin.

(a) The value $s(10)$ is the difference in the odometer readings from $t = 0$ to $t = 10$. Explain.
(b) The value $s'(t)$ is the speedometer reading at time t. Explain.
(c) The value $y'(t)$ is the rate of change in altitude, while $x'(t)$ is the rate of horizontal approach to the resort. Explain.
(d) What is the average rate of vertical ascent? What is the average speed for the trip?

33. A child walks with speed k from the center of a merry-go-round to its edge, while the equipment rotates counterclockwise with constant angular speed ω. The motion of the child relative to the ground is $x = kt\cos\omega t$, $y = kt\sin\omega t$.
(a) Find the *velocities* $\dot{x} = dx/dt$, $\dot{y} = dy/dt$.
(b) Determine the *speed*.
(c) The child experiences a *Coriolis* force opposite to the direction of rotation, tangent to the edge of the merry-go-round. The magnitude of this force is the mass m of the child times the factor $\sqrt{\ddot{x}(0)^2 + \ddot{y}(0)^2}$, where $\ddot{x} = d^2x/dt^2$. Find this force.

★34. (a) Find a parametric curve $x = f(t)$, $y = g(t)$ passing through the points $(1,1)$, $(2,2)$, $(4,2)$, $(5,1)$, $(3,0)$, and $(1,1)$ such that the functions f and g are both piecewise linear and the curve is a polygon whose vertices are the given points in the given order.
(b) Compute the length of this curve by formula (4) and then by elementary geometry. Compare the results.
(c) What is the area of the surface obtained by revolving the given curve about the y axis?

★35. At each point (x_0, y_0) of the parabola $y = x^2$, the tangent line is drawn and a point is marked on this line at a distance of 1 unit from (x_0, y_0) to the right of (x_0, y_0).
(a) Describe the collection of points thus obtained as a parametrized curve.
(b) Describe the collection of points thus obtained in terms of a relation between x and y.

★36. If $x = t$ and $y = g(t)$, show that the points where the speed is maximized are points of inflection of $y = g(x)$.

★37.[5] (a) Looking at a map of the United States, estimate the length of the coastline of Maine.
(b) Estimate the same length by looking at a map of Maine.
(c) Suppose that you used detailed local maps to compute the length of the coastline of Maine. How would the results compare with that obtained in part (b)?
(d) What is the "true" length of the coastline of Maine?
(e) What length for the coastline can you find given in an atlas or almanac?

★38. On a movie set, an auto races down a street. A follow-spot lights the action from 20 meters away, keeping a constant distance from the auto in order to maintain the same reflected light intensity for the camera. The follow-spot location (x, y) is the *pursuit curve*
$$x = t - 20\,\mathrm{sech}\left(\frac{t}{20}\right), \qquad y = 20\,\mathrm{sech}\left(\frac{t}{20}\right),$$
called a *tractrix*. Graph it.

[5] For further information on the ideas in this exercise, see B. Mandelbrot, *Fractals: Form, Chance and Dimension*, Freeman, New York (1977).

10.5 Length and Area in Polar Coordinates

Some length and area problems are most easily solved in polar coordinates.

The formula $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$ for the length of a parametric curve can be applied to the curve $r = f(\theta)$ in polar coordinates if we take the parameter to be θ in place of t. We write:

$$x = r\cos\theta = f(\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = f(\theta)\sin\theta.$$

Suppose that θ runs from α to β (see Fig. 10.5.1). By formula (4) of Section 10.4, the length is

$$\int_\alpha^\beta \sqrt{[f'(\theta)\cos\theta - f(\theta)\sin\theta]^2 + [f'(\theta)\sin\theta + f(\theta)\cos\theta]^2}\, d\theta$$

which simplifies to

$$\int_\alpha^\beta \sqrt{[f'(\theta)]^2 + [f(\theta)]^2}\, d\theta.$$

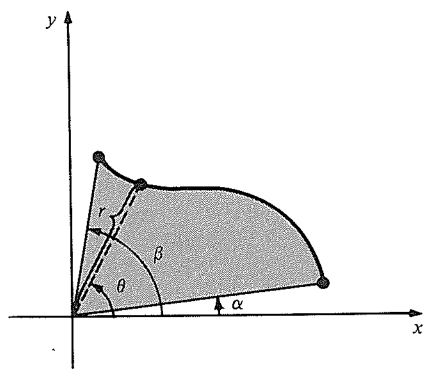


Figure 10.5.1. The length of the curve is $\int_\alpha^\beta \sqrt{(dr/d\theta)^2 + r^2}\, d\theta$.

Arc Length in Polar Coordinates

The length of the curve $r = f(\theta)$, $\alpha \leqslant \theta \leqslant \beta$, is given by

$$L = \int_\alpha^\beta \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta. \tag{1}$$

One can obtain the same formula by an infinitesmal argument, following Fig. 10.5.2. By Pythagoras' theorem, $ds^2 = dr^2 + (r\,d\theta)^2$, or $ds = \sqrt{dr^2 + r^2\,d\theta^2}$. If we use $dr = f'(\theta)\,d\theta$, this becomes

$$ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 d\theta^2 + r^2\,d\theta^2} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta,$$

so

$$L = \int_\alpha^\beta ds = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta.$$

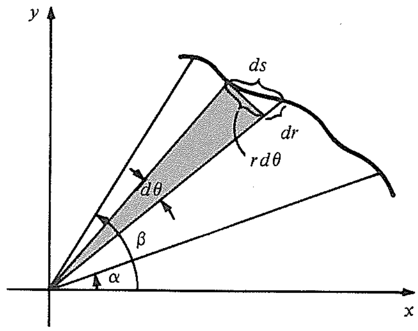


Figure 10.5.2. The infinitesimal element of arc length ds equals $\sqrt{dr^2 + r^2\,d\theta^2}$.

Example 1 Find the length of the curve $r = 1 - \cos\theta$, $0 \leqslant \theta \leqslant 2\pi$.

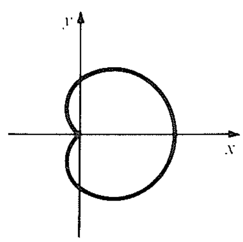
Solution We find $dr/d\theta = \sin\theta$, so by equation (1),

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\sin^2\theta + (1-\cos\theta)^2}\,d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\,d\theta \\
&= \sqrt{2}\int_0^{2\pi}\sqrt{1-\cos\theta}\,d\theta = \sqrt{2}\int_0^{2\pi}\sqrt{2\sin^2\frac{\theta}{2}}\,d\theta \\
&= 2\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta = 4\int_0^{\pi}\sin u\,du \qquad \left(u = \frac{\theta}{2}\right) \\
&= 4(-\cos u)\Big|_0^{\pi} = 8. \ \blacktriangle
\end{aligned}
$$

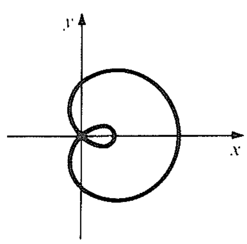
Example 2 Find the length of the cardioid $r = 1 + \cos\theta$ $(0 \leqslant \theta \leqslant 2\pi)$.

Solution (This curve is sketched in the accompanying figure.) The length is

$$
\begin{aligned}
L &= \int_0^{2\pi}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^{2\pi}\sqrt{(1+\cos\theta)^2 + \sin^2\theta}\,d\theta \\
&= \int_0^{2\pi}\sqrt{2 + 2\cos\theta}\,d\theta.
\end{aligned}
$$



(a)



(b)

(a) $r = 1 + \cos\theta$ ("cardioid");
(b) $r = 1 + 2\cos\theta$ ("limaçon").

This can be simplified, by the half-angle formula $\cos^2(\theta/2) = (1+\cos\theta)/2$, to

$$L = \int_0^{2\pi} 2\cos\frac{\theta}{2}\,d\theta = 0.$$

Something is wrong here! We forgot that $\cos(\theta/2)$ can be negative, while the square root $\sqrt{2 + 2\cos\theta}$ must be positive; i.e.,

$$\sqrt{2+2\cos\theta} = \sqrt{4\cos^2\frac{\theta}{2}} = 2\left|\cos\frac{\theta}{2}\right|.$$

The correct evaluation of L is as follows:

$$L = \int_0^{2\pi} 2\left|\cos\frac{\theta}{2}\right| d\theta = \int_0^{\pi} 2\cos\frac{\theta}{2}\,d\theta - \int_{\pi}^{2\pi} 2\cos\frac{\theta}{2}\,d\theta$$

since $\cos(\theta/2) > 0$ on $(0,\pi)$ and $\cos(\theta/2) < 0$ on $(\pi, 2\pi)$. Thus

$$L = 4\sin\frac{\theta}{2}\Big|_0^{\pi} - 4\sin\frac{\theta}{2}\Big|_{\pi}^{2\pi} = 4(1-0) - 4(0-1) = 8. \ \blacktriangle$$

The curve expressed in polar coordinates by the equation $r = f(\theta)$, together with the rays $\theta = \alpha$ and $\theta = \beta$, encloses a region of the type shown (shaded) in Fig. 10.5.3. We call this the region *inside* the graph of f on $[\alpha, \beta]$.

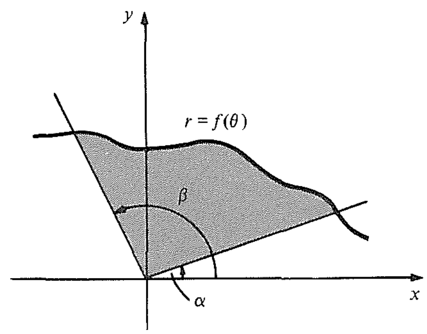


Figure 10.5.3. The region inside the graph $r = f(\theta)$ on $[\alpha, \beta]$ is shaded.

We wish to find a formula for the area of such a region as an integral involving the function f. We begin with the simplest case, in which f is a constant function $f(\theta) = k$. The region inside the curve $r = k$ on $[\alpha, \beta]$ is then a circular sector with radius k and angle $\beta - \alpha$ (see Fig. 10.5.4). The area is $(\beta - \alpha)/2\pi$ times the area πk^2 of a circle of radius k, or $\frac{1}{2}k^2(\beta - \alpha)$. We can express this as the integral $\int_\alpha^\beta \frac{1}{2} f(\theta)^2 \, d\theta$.

If f is a step function, with $f(\theta) = k_i$, on (θ_{i-1}, θ_i), then the region inside the graph of f is of the type shown in Fig. 10.5.5. Its area is equal to the sum of the areas of the individual sectors, or

$$\sum_{i=1}^{n} \frac{1}{2} k_i^2 \Delta\theta_i = \int_\alpha^\beta \frac{1}{2} [f(\theta)]^2 \, d\theta.$$

By approximating f with step functions, we conclude that the same formula holds for general f.

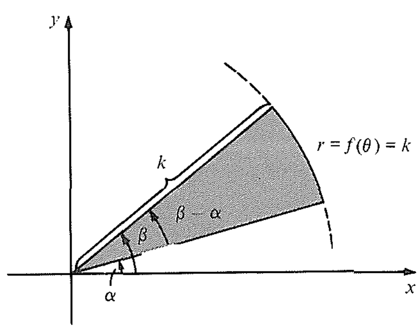


Figure 10.5.4. The area of the sector is $\frac{1}{2}k^2(\beta - \alpha)$.

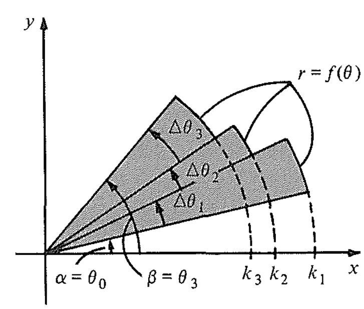


Figure 10.5.5. The area of the shaded region is $\sum \frac{1}{2} k_i^2 \Delta\theta_i$.

Area in Polar Coordinates

The area of the region enclosed by the curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2} \int_\alpha^\beta f(\theta)^2 \, d\theta = \frac{1}{2} \int_\alpha^\beta r^2 \, d\theta.$$

This formula can also be obtained by an infinitesimal argument. Indeed, the area dA of the shaded triangle in Fig. 10.5.2 is $\frac{1}{2}(\text{base}) \times (\text{height})$

$= \frac{1}{2}(r\,d\theta)r = \frac{1}{2}r^2\,d\theta$, so the area inside the curve is

$$\int_\alpha^\beta dA = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$

which agrees with the formula in the preceding box.

Example 3 Find the area enclosed by one petal of the four-petaled rose $r = \cos 2\theta$ (see Fig. 5.6.3).

Solution The petal shown in Fig. 10.5.6 is enclosed by the arc $r = \cos 2\theta$ and the rays $\theta = -\pi/4$ and $\theta = \pi/4$. Notice that the rays do not actually appear in the boundary of the figure, since the radius $r = \cos(\pm\pi/2)$ is zero there. The area is given by $\frac{1}{2}\int_{-\pi/4}^{\pi/4} r^2\,d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4}\cos^2 2\theta\,d\theta$. By the half-angle formula this is

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4}\frac{1+\cos 4\theta}{2}\,d\theta = \frac{1}{4}\left(\theta + \frac{\sin 4\theta}{4}\right)\Big|_{-\pi/4}^{\pi/4} = \frac{\pi}{8}. \blacktriangle$$

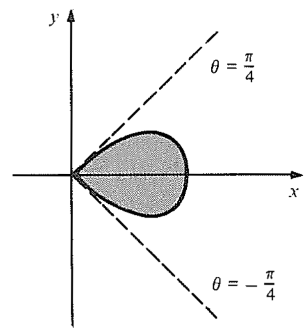


Figure 10.5.6. One leaf of the four-petaled rose $r = \cos 2\theta$.

Example 4 Find the area enclosed by the cardioid $r = 1 + \cos\theta$ (see Fig. 5.6.6).

Solution The area enclosed is defined by $r = 1 + \cos\theta$ and the full range $0 \leqslant \theta \leqslant 2\pi$, so

$$A = \frac{1}{2}\int_0^{2\pi}(1+\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(1 + 2\cos\theta + \cos^2\theta)\,d\theta.$$

Again using the half-angle formula,

$$A = \frac{1}{2}\int_0^{2\pi}\left(\frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left[\frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{4}\right]\Big|_0^{2\pi} = \frac{3\pi}{2}. \blacktriangle$$

Example 5 Find a formula for the area between two curves in polar coordinates.

Solution Suppose $r = f(\theta)$ and $r = g(\theta)$ are the two curves with $f(\theta) \geqslant g(\theta) > 0$. We are required to find a formula for the shaded area in Fig. 10.5.7. The area is just the difference betweeen the areas for f and g; that is,

$$A = \frac{1}{2}\int_\alpha^\beta\left[f(\theta)^2 - g(\theta)^2\right]d\theta. \blacktriangle$$

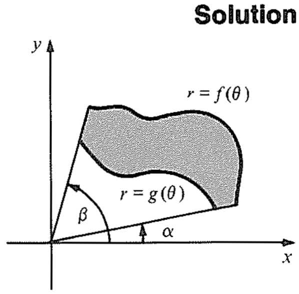
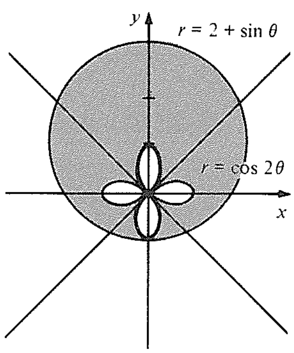


Figure 10.5.7. The area of the shaded region is $\frac{1}{2}\int_\alpha^\beta[f(\theta)^2 - g(\theta)^2]\,d\theta$.

Example 6 Sketch and find the area of the region between the curves $r = \cos 2\theta$ and $r = 2 + \sin\theta$, $0 \leqslant \theta \leqslant 2\pi$.

Solution The curves are sketched in Fig. 10.5.8. To do this, we plotted points for θ at multiples of $\pi/4$ and then noted whether r was increasing or decreasing on each of the intervals between these θ values. To find the shaded area, we must be careful because of the sign changes of $g(\theta) = \cos 2\theta$. The inner loops are described in the following way by positive functions:



$$g(\theta) = \begin{cases} \cos 2\theta, & 0 \leqslant \theta \leqslant \frac{\pi}{4} \\ -\cos 2\theta, & \frac{\pi}{4} \leqslant \theta \leqslant \frac{3\pi}{4} \\ \cos 2\theta, & \frac{3\pi}{4} \leqslant \theta \leqslant \frac{5\pi}{4} \\ -\cos 2\theta, & \frac{5\pi}{4} \leqslant \theta \leqslant \frac{7\pi}{4} \\ \cos 2\theta, & \frac{7\pi}{4} \leqslant \theta \leqslant 2\pi \end{cases}.$$

In fact, we are lucky because in the formula in Example 5, $g(\theta)$ is squared anyway, so the shaded area is simply

$$\frac{1}{2}\int_0^{2\pi}\left[(2+\sin\theta)^2 - \cos^2 2\theta\right]d\theta$$
$$= \frac{1}{2}\left[\int_0^{2\pi} 4\,d\theta + \int_0^{2\pi} 4\sin\theta\,d\theta + \int_0^{2\pi}\sin^2\theta\,d\theta - \int_0^{2\pi}\cos^2 2\theta\,d\theta\right]$$
$$= \frac{1}{2}\left[8\pi + 0 + \pi - \pi\right] = 4\pi. \blacktriangle$$

Exercises for Section 10.5

Find the length of the curves in Exercises 1–4.

1. $r = 3(1+\sin\theta)$; $0 \leqslant \theta \leqslant 2\pi$.
2. $r = 1/(\cos\theta + \sin\theta)$; $0 \leqslant \theta \leqslant \pi/2$.
3. $r = 4\theta^2$; $0 \leqslant \theta \leqslant 3$.
4. $r = 8\theta^2$; $0 \leqslant \theta \leqslant 1$.

Sketch and find the area of the region bounded by the curves in Exercises 5–10.

5. $r = 3\sin\theta$; $0 \leqslant \theta \leqslant \pi$.
6. $r = 2(1+\sin\theta)$; $0 \leqslant \theta \leqslant 2\pi$.
7. $r = \theta$; $0 \leqslant \theta \leqslant 3\pi/2$.
8. $r = \theta\cos(\theta^3)$; $0 \leqslant \theta \leqslant \pi/4$.
9. $r = 4 + \sin\theta$; $0 \leqslant \theta \leqslant 2\pi$.
10. $r = \theta + \sin 4\theta$; $\pi/4 \leqslant \theta \leqslant \pi$. [*Hint*: Find the critical points of r.]
11. Check the arc length formula in polar coordinates for a circle.
12. Check the area formula in polar coordinates for a segment of a circle and a whole circle.

In Exercises 13–16, sketch and find the length (as an integral) of the graph of $r = f(\theta)$, $\alpha \leqslant \theta \leqslant \beta$. (The answer may be in the form of an integral.) Then find the area of the region bounded by this graph and the rays $\theta = \alpha$ and $\theta = \beta$.

13. $r = \tan(\theta/2)$; $-\pi/2 \leqslant \theta \leqslant \pi/2$.
14. $r = \theta + \sin(\theta^2)$; $-\pi/4 \leqslant \theta \leqslant 3\pi/4$.
15. $r = \sec\theta + 2$; $0 \leqslant \theta \leqslant \pi/4$.
16. $r = 2e^{3\theta}$; $\ln 2 \leqslant \theta \leqslant \ln 3$.

In Exercises 17–20, find the length of and areas bounded by the following curves between the rays indicated. Express the areas as numbers but leave the length as integrals.

17. $r = \theta(1+\cos\theta)$; $\theta = 0$, $\theta = \pi/2$.
18. $r = 1/\theta$; $\theta = 1$, $\theta = \pi$.
19. $f(\theta) = \sqrt{1 + 2\sin 2\theta}$; $\theta = 0$, $\theta = \pi/2$.
20. $f(\theta) = \theta^2 - (\pi/2)\theta + 4$; $\theta = 0$, $\theta = \pi/2$.

In Exercises 21–24, sketch and find the area of each of the regions between each of the following pairs of curves $(0 \leqslant \theta \leqslant 2\pi)$. Then find the length of the curves which bound the regions.

21. $r = \cos\theta$, $r = \sqrt{3}\sin\theta$.
22. $r = 3$, $r = 2(1+\cos\theta)$.
23. $r = 2\cos\theta$, $r = 1 + \cos\theta$.
24. $r = 1$, $r = 1 + \cos\theta$.
25. The curve $r = e^\theta$ is called a *logarithmic spiral*. Find the length of the loop of the logarithmic spiral for θ in $[2n\pi, 2(n+1)\pi]$.

26. Suppose that the distance from the origin to $(x, y) = (f(t), g(t))$ attains its maximum value at $t = t_0$. Show that the tangent line at t_0 is perpendicular to the line from the origin to the point $(f(t_0), g(t_0))$.

★27. An elliptical orbit is parametrized by $x = a\cos\theta$, $y = b\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$. This parametrization is 2π-periodic. In Chapter 18 we shall show that for any T-periodic parametrization of a continuously differentiable closed curve $x = x(t)$, $y = y(t)$ which is a *simple* (never crosses itself),

$$\text{area enclosed} = \int_0^T \frac{1}{2}[x(t)\dot{y}(t) - \dot{x}(t)y(t)]\,dt,$$

where $\dot{x}(t) = dx/dt$ and $\dot{y}(t) = dy/dt$. (See also Review Exercise 95 for this Chapter.)

(a) Use this formula to verify that the area enclosed by an ellipse of semiaxes a and b is πab.

(b) Apply the formula to the case of a curve $x(t) = r\cos t$, $y(t) = r\sin t$, where $r = r(t)$, showing that the area enclosed is $\frac{1}{2}\int_0^T r^2\,dt$.

Review Exercises for Chapter 10

Evaluate the integrals in Exercises 1–50.

1. $\int 3\sin^2 x\cos x\,dx$
2. $\int \sin^2 2x\cos^3 2x\,dx$
3. $\int \sin 3x\cos 5x\,dx$
4. $\int \cos 4x\sin 6x\,dx$
5. $\int \frac{x^3}{\sqrt{1-x^2}}\,dx \quad (|x| < 1)$
6. $\int \frac{dx}{(x^2+2)^2}$
7. $\int \frac{\sqrt{x^2-16}}{x}\,dx \quad (x > 4)$
8. $\int \frac{dx}{\sqrt{x^2-16}} \quad (x > 4)$
9. $\int \frac{dx}{x^2+x+2}$
10. $\int \frac{dx}{\sqrt{x^2+x+2}}$
11. $\int \frac{dx}{x^3+x^2}$
12. $\int \frac{dx}{x^3-27}$
13. $\int \frac{x^3}{(x^2+1)^2}\,dx$
14. $\int \frac{x^2}{(x+1)^3}\,dx$
15. $\int \frac{dx}{x^2+4x+5}$
16. $\int \sec^6\theta\,d\theta$
17. $\int \sin\sqrt{x}\,dx$
18. $\int \frac{dx}{1+\cos ax}$
19. $\int \frac{dx}{(1-\cos ax)^2}$
20. $\int \frac{dx}{1-x^4}$
21. $\int \frac{\sin^2 x}{\cos x}\,dx$
22. $\int \ln\left(\frac{x+a}{x-a}\right)dx$
23. $\int \frac{\tan^{-1}x}{1+x^2}\,dx$
24. $\int \frac{dx}{x^4+1}$
25. $\int \frac{x}{x^3-9}\,dx$
26. $\int (x+5)\ln x\,dx$
27. $\int e^{\sqrt{x}}\,dx$
28. $\int x^3\sqrt{1-x^2}\,dx$
29. $\int \frac{dx}{1+e^x}$
30. $\int \frac{x}{(x-3)^8}\,dx$
31. $\int \left(\frac{x}{x^2-1}\right)^3 dx$
32. $\int \sqrt{x^2+2x+3}\,dx$
33. $\int \sin 3x\cos 2x\,dx$
34. $\int \sin^2 3x\cos^4 3x\,dx$
35. $\int \frac{x}{x^2+1}\,dx$
36. $\int \frac{x}{(x^2+1)^2}\,dx$
37. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$
38. $\int (e^x+1)^3 e^x\,dx$
39. $\int_2^3 \frac{x}{x^2+1}\,dx$
40. $\int_0^{\pi/2} \sin x\, e^{\cos x}\,dx$
41. $\int \frac{x}{x^2+3}\,dx$
42. $\int \frac{x}{\sqrt{x+1}}\,dx$
43. $\int x^3\ln x\,dx$
44. $\int \sqrt{1+\sin x}\cdot\cos x\,dx$
45. $\int_1^2 \frac{(\ln 3x+5)^3}{x}\,dx$
46. $\int_e^{e^4} \frac{\ln t^2}{t^2}\,dt$
47. $\int_0^1 \sinh^2 x\,dx$
48. $\int_{\pi/8}^{\pi/4} \frac{\sin\theta}{\sqrt{1-\cos^2\theta}}\,d\theta$
49. $\int_0^{2\pi} \frac{\sin\theta}{1+\cos\theta+\cos^2\theta}\,d\theta$
50. $\int_0^{\pi/100} \sec^2 100x\,dx$

In Exercises 51–54, find the length of the given graph.

51. $y = 3x^{3/2}$, $0 \leqslant x \leqslant 9$
52. $y = (x+1)^{3/2} + 1$, $0 \leqslant x \leqslant 2$.
53. $y = \frac{x^3}{3} + \frac{1}{4x}$, $1 \leqslant x \leqslant 2$.
54. $y = \frac{x^4}{4} + \frac{1}{8x^2}$, $1 \leqslant x \leqslant 2$.

In Exercises 55–58, find the area of the surface obtained by revolving the given graph about the given axis.

55. $y = x^2$, $0 \leqslant x \leqslant 1$, about the y axis.
56. $y = \sqrt{x}$, $0 \leqslant x \leqslant 1$, about the x axis.
57. $y = \log_{10}x$, $10 \leqslant x \leqslant 100$, about the y axis.
58. $y = 2^x$, $3 \leqslant x \leqslant 4$, about the x axis.

For each of the pairs of parametric equations in Exercises 59–64, sketch the curve and find an equation in x and y by eliminating the parameter.

59. $x = t^2;\ y = t - 1$
60. $x = 2t + 5;\ y = t^3$
61. $x = 3t;\ y = 2t + 1$
62. $x = t;\ y = t$
63. $x = 0;\ y = t^4$
64. $x = t^2\sqrt{t^2 - 1}\,;\ y = t^2$
65. Find the equation of the tangent line to the curve $x = t^4, y = 1 + t^3$ at $t = 1$.
66. Find the equation of the tangent line to the parametric curve $x = 3\cos t, y = \sin t$ at $t = \pi/4$.
67. Find the arc length of $x = t^2, y = 2t^4$ from $t = 0$ to $t = 2$.
68. Find the length of $x = e^t\sin t,\ y = e^t\cos t$ from $t = 0$ to $t = \pi/2$.

Find the arc length (as an integral if necessary) and area enclosed by each of the graphs given in polar coordinates in Exercises 69–74.

69. $r = \theta^2;\ 0 \leqslant \theta \leqslant \dfrac{\pi}{2}$.
70. $r = \dfrac{1}{\cos\theta}\,;\ 0 \leqslant \theta \leqslant \dfrac{\pi}{4}$.
71. $r = \frac{1}{2} + \cos 2\theta;\ 0 \leqslant \theta \leqslant \pi$.
72. $r = 2|\cos\theta|;\ 0 \leqslant \theta \leqslant 2\pi$.
73. $r = 3\cos^4\dfrac{\theta}{4}\,;\ 0 \leqslant \theta \leqslant \pi$.
74. $r = \dfrac{1}{2}\sin^2\dfrac{\theta}{2}\,;\ \dfrac{\pi}{4} \leqslant \theta \leqslant \dfrac{3\pi}{4}$.

If f is a function on $[0, 2\pi]$, then the numbers

$$\left.\begin{aligned} a_m &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx\,dx \\ b_m &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx\,dx \end{aligned}\right\}\qquad (m = 0, 1, 2, \dots)$$

are called the *Fourier coefficients* of f. Find all the Fourier coefficients of each of the functions in Exercises 75–82.

75. $\sin 2x$
76. $\sin 5x$
77. $\cos 3x$
78. $\cos 8x$
79. $3\cos 4x$
80. $2\cos 8x + \sin 7x + \cos 9x$
81. $\sin^2 x$
82. $\cos^3 x$
83. The solution of the *logistic equation* of population biology, $dN/dt = (k_1N - k_2)N$, $N(0) = N_0$, requires the evaluation of the definite integral
$$\int_{N_0}^{N(t)} \frac{du}{(k_1u - k_2)u}\,.$$
(a) Evaluate by means of partial fraction methods and compare your answer with Exercise 19, Section 8.5.
(b) The integral is just the time t. Solve for $N(t)$ in terms of t, using exponentials.
(c) Find $\lim_{t\to\infty} N(t)$ when it exists.
84. Kepler's second law of planetary motion says that *the radial segment drawn from the sun to a planet sweeps out equal areas in equal times*. Locate the origin $(0, 0)$ at the sun and introduce polar coordinates (γ, θ) for the planet location. Assume the *angular momentum* of the planet (of mass m) about the sun is constant; $mr^2\dot\theta = mk$, k = constant, and $\dot\theta = d\theta/dt$. Establish Kepler's second law by showing $\int_s^{s+h} r^2\dot\theta\,dt$ is the same for all times s; thus the area swept out is the same for all time intervals of length h.
85. An elliptical satellite circuits the earth in a circular orbit. The angle ϕ between its major axis and the direction to the earth's center oscillates between $+\phi_m$ and $-\phi_m$ (*librations* of the earth satellite). It is assumed that $0 < \phi_m < \pi/2$, so that the satellite does not tumble end over end. The time T for one complete cycle of this oscillation is given by
$$T = \frac{4}{\pi}\int_0^{\phi_m} \frac{d\phi}{\sqrt{\cos 2\phi - \cos 2\phi_m}}\,.$$
Change variables in the integral via the formulas
$$\sin\phi = \sin\phi_m \sin\beta \text{ (which defines } \beta\text{)},$$
$$\cos 2\phi = 1 - 2\sin^2\phi,$$
$$\cos 2\phi_m = 1 - 2\sin^2\phi_m\,,$$
to obtain the *elliptic integral representation*
$$T = \frac{4}{\pi\sqrt{2}}\int_0^{\pi/2} \frac{d\beta}{\sqrt{1 - k^2\sin^2\beta}}$$
for the period of libration T, where $k^2 = \sin^2\phi_m$.

★86. An engineer is studying the impact of an infinite bar by a short round-headed bar, making a maximum indentation α_1. Applying Hertz' theory of impact, she obtains the equation $\frac{1}{2}\rho c_0\Omega\alpha' = k(\alpha_1^{3/2} - \alpha^{3/2})$ for the indentation α at time t. The symbols ρ, c_0, Ω, k are constants. The equation is solved by an initial integration to get
$$t = \frac{\rho c_0\Omega}{2k\sqrt{\alpha_1}}\int_0^{\alpha/\alpha_1} \frac{du}{1 - u^{3/2}}\,.$$
(a) Evaluate the integral by making the substitution $v = \sqrt{u}$, followed by the method of partial fractions.
(b) Substitute $s = (4tk\sqrt{\alpha_1})/(3\rho c_0\Omega)$ to obtain
$$9s = \frac{2\pi}{\sqrt{3}} + 2\ln\left|\frac{1 + y + y^2}{(1 - y)^2}\right| - 4\sqrt{3}\tan^{-1}\left(\frac{2y + 1}{\sqrt{3}}\right),$$
where $y = \sqrt{\alpha/\alpha_1}$.

★87. Find a general formula for $\int dx/\sqrt{ax^2 + bx + c}\,$; $a \neq 0$. There will be two cases, depending upon the sign of a.

★88. Let $f(x) = x^n$, $0 < a \leqslant x \leqslant b$. For which rational values of n can you evaluate the integral occurring in the formula for:
(a) The area under the graph of f?
(b) The length of the graph of f?
(c) The volume of the surface obtained by revolving the region under the graph of f about the x axis? The y axis?

(d) The area of the surface of revolution obtained by revolving the graph of f about the x axis? The y axis?

Evaluate these integrals.

★89. Same as Exercise 88, but with $f(x) = 1 + x^n$.

★90. Same as Exercise 88, but with $f(x) = (1 + x^2)^n$.

★91. (a) Find the formula for the area of the surface obtained by revolving the graph of $r = f(\theta)$ about the x axis, $\alpha \leqslant \theta \leqslant \beta$.
(b) Find the area of the surface obtained by revolving $r = \cos 2\theta$, $-\pi/4 \leqslant \theta \leqslant \pi/4$ about the x axis (express as an integral if necessary).

★92. Consider the integral

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

(a) Show that, for $k = 0$ and $k = 1$, this integral can be evaluated in terms of trigonometric and exponential functions and their inverses.

(b) Show that, for any k, the integral may be transformed to one of the form

$$\int \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

(This integrand occurs in the sunshine formula—see the supplement to Section 9.5.)

(c) Show that the integral

$$\int \sqrt{1-k^2\sin^2\theta}\; d\theta$$

(which also occurs in the sunshine formula) arises when one tries to find the arc length of an ellipse $x^2/a^2 + y^2/b^2 = 1$. Express k in terms of a and b.

Due to the result of part (c), the integrals in parts (a), (b), and (c) are called *elliptic integrals*.

★93. Consider the parametric curve given by $x = \cos mt$, $y = \sin nt$, when m and n are integers. Such a curve is called a *Lissajous figure* (see Example 6, Section 10.4).

(a) Plot the curve for $m = 1$ and $n = 1, 2, 3, 4$.

(b) Describe the general behavior of the curve if $m = 1$, for any value of n. Does it matter whether n is even or odd?

(c) Plot the curve for $m = 2$ and $n = 1, 2, 3, 4, 5$.

(d) Plot the curve for $m = 3$ and $n = 4, 5$.

★94. (Lissajous figures continued). The path $x = x(t)$, $y = y(t)$ of movement of the *tri-suspension pendulum* of Fig. 10.R.1 produces a Lissajous figure of the general form $x = A_1\cos(\omega_1 t + \theta_1)$, $y = A_2\sin(\omega_2 t + \theta_0)$.

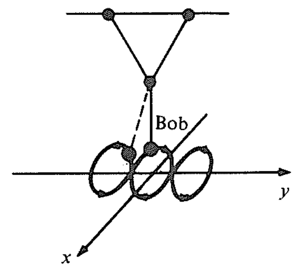


Figure 10.R.1. The bob on this pendulum traces out a Lissajous figure.

(a) Draw the Lissajous figures for $\omega_1 = \omega_2 = 1$, $A_1 = A_2$, for some sample values of θ_1, θ_2. The figures should come out to be straight lines, circles, ellipses.

(b) When $\omega_1 = 1$, $\omega_2 = 3$, $A_1 = A_2 = 1$, the bob retraces its path, but has two self-intersections. Verify this using the results of Exercise 93. Conjecture what happens when ω_2/ω_1 is the ratio of integers.

(c) When $\omega_2/\omega_1 = \pi$, $A_1 = A_2 = 1$, the bob does not retrace its path, and has infinitely many self-intersections. Verify this, graphically. Conjecture what happens when ω_2/ω_1 is irrational (not the quotient of integers).

★95. Consider the curve $r = f(\theta)$ for $0 \leqslant \theta \leqslant 2\pi$ as a parametric curve: $x = f(t)\cos t$, $y = f(t)\sin t$. Assuming that $f(\theta) > 0$ for all θ in $[0, 2\pi]$ and that $f(2\pi) = f(0)$, show that the area enclosed by the curve is given by

$$-\int_0^{2\pi} y\frac{dx}{dt}\,dt \tag{A}$$

as well as by $\int_0^{2\pi} x(dy/dt)\,dt$ and by the more symmetric formula

$$\frac{1}{2}\int_0^{2\pi}\left[x\frac{dy}{dt} - y\frac{dx}{dt}\right]dt.$$

[*Hint*: Substitute the definitions of x and y into (A), integrate by parts, and use the formula for area in polar coordinates.] These formulas are in fact valid for any closed parametric curve. (See Section 18.4.)

★96. (a) If r is a non-repeated root of $Q(x)$, show that the portion of the partial fraction expansion of $P(x)/Q(x)$ corresponding to the factor $x - r$ is $A/(x - r)$ where $A = P(r)/Q'(r)$. (b) Use (a) to calculate $\int [(x^2 + 2)/(x^3 - 6x + 11x - 6)]\,dx$.